Prove $6 \mid n^3 + 5n$ $(n \geq 0)$

$\exists k \in \mathbb{Z}$:

$n^3 + 5n = 6k$

Want to show: $(n+1)^3 + 5(n+1) = 6j$ $(j \in \mathbb{Z})$

Show

$n^3 + 3n^2 + 3n + 1 + 5n + 5 = 6j$

Show $3n^2 + 3n + 1 + 5 = 6j$

Show $3(n^2+n) + \underline{6} = 6j$

Show $3(n^2+n) = 6j$

Show n^2+n is even

if $n = 2i$

$n^2+n = 4i^2 + 2i$

$= 2(2i^2 + i)$

if $n = 2i+1$

$n^2+n = 4i^2 + 4i + 1 + 2i + 1$

$= 2(2i^2 + 3i + 1)$?

The Sum Rule

If a task T can be done by either doing T_1 or T_2,
and T_1 can be done n_1 ways,
and T_2 ~~~~~~~ n_2 ~,
the T can be done $n_1 + n_2$ ways.

Ex: Choose a lunch! Soup or Salad

$\downarrow$ 3 $\quad$ $\downarrow$ 5

$3 + 5$

A variable is a letter or a letter + a digit

T_1: pick a variable which is just a letter

T_2: ~ a letter + a digit

n_1: 26

n_2: pick a letter : 26

and then

pick a digit : 10

$26 \cdot 10 = 260$

n_2 : 260

$26 + 260 = 286$

SUBTRACTION

"Principle of Inclusion/Exclusion"

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

Ex: How many 8-bit strings start w/1 or end w/0?

11111111

00000000

00001110

11101011

$A: \{1\ \_\ \_\ \_\ \_\ \_\ \_\ \_\}$

$B: \{\_\ \_\ \_\ \_\ \_\ \_\ \_\ 0\}$

$A \cap B: \{1\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}\ 0\}$

$|A| = 128$

$|B| = 128$

$|A \cap B| = 64$

$|A \cup B| = 128 + 128 - 64 = \cancel{192}\ 192$

SUBTRACTION

"Principle of Inclusion/Exclusion"

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

Ex: How many 8-bit strings start w/ 1 or end w/ 0?

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A: { 1 _ _ _ _ _ _ _ }

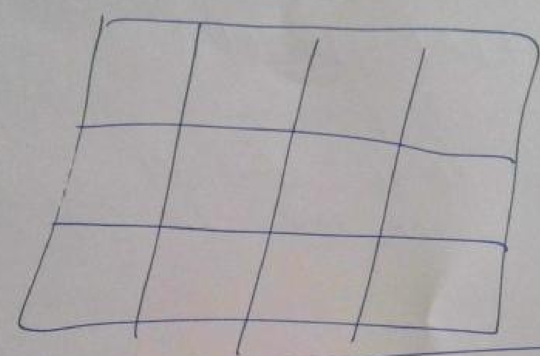
B: { _ _ _ _ _ _ _ 0 }

$|A| = 128$
$|B| = 128$
$|A \cap B| = 64$

$A \cap B$: { 1 $\underbrace{\_\ \_\ \_\ \_\ \_\ \_}$ 0 }

$|A \cup B| = 128 + 128 - 64 = \cancel{192}\ 192$

The Pigeonhole Principle



If k is a positive integer and $k+1$ (or more) objects are placed into k boxes, then at least one box has 2 or more objects in it.

Example: In a room with 27 people, at least two have the same first initial.

Make 26 boxes

A B C . . . Y Z

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10 pt quiz: How many students need ~~to th~~ take the quiz in order to <u>guarantee</u> that at least 2 have the same score?

11 possible scores

<u>12 students</u>

0 1 2 . . . 9 10

11 boxes

In any group of 8 people, at least 2 were born on the same day of the week.

---

fact: Every positive integer has a multiple of itself that can be written with only 1's & 0's.

ex: $1 = 1$

$5 \times 2 = 10$

$37 \times 3 = 111$

$25 \times 4 = 100$

$20 \times 5 = 100$

$37 \times 5 \times 6 = 1110$

185×6

$7 = ?$

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185×6

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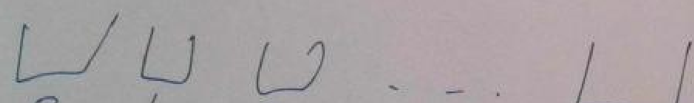
given pos. int n

Write the following $n+1$ ints:

$$\underbrace{1,\ 11,\ 111,\ 1111,\ 11111,\ \ldots,\ 1111111\ldots1}_{n+1}$$

Place each into a box labeled $0, 1, 2, \ldots, n-1$ based on congruence (mod n)



$$\underbrace{0 \quad 1 \quad 2 \quad \ldots \quad n-1}_{n \text{ boxes}}$$

$$\begin{array}{r} 111111111 \\ -\ \ 1111111 \\ \hline 110000000 \end{array}$$

PERMUTATIONS

A **permutation** is an **ordered** arrangement of objects.

An **r-permutation** is an ordered arrangement of r objects.

Example: given the letters A, B, C,

CAB is a permutation

BA is a 2-permutation

Ex: 5 students, pick 3 to place on stage, in 3 positions

Task T: pick the 3 students & where they stand.

T_1 : pick student for pos. 1 $\quad n_1 = 5$

T_2 : ~ 2 $\quad n_2 = 4$

T_3 : ~ 3 $\quad n_3 = 3$

$5 \cdot 4 \cdot 3 = \underline{60}$

n objects: choose r, order matters

T_1 : choose object 1 $\quad$ n ways

T_2 : ~ 2 $\quad$ $n-1$

$\vdots$ $\quad$ $n-2$

$\vdots$

T_r : ~ r $\quad$ $n-(r-1)$

$$n(n-1)(n-2)\cdots(n-(r-1))$$

$P(n,r)$ means "the number of r-permutations of n objects"

$$= n(n-1)(n-2)\cdots(n-r+1)$$

$P(11,6)$

$$= \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{11!}{5!}$$

$$P(7,5) = \cancel{\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1} = \frac{7!}{4!}}$$

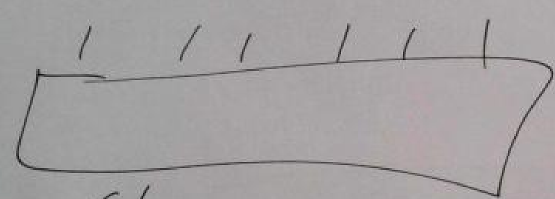
$$\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1} = \frac{7!}{2!}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

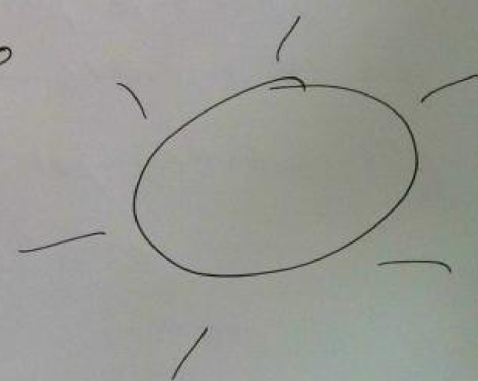
Ex: 10 runners; 3 trophies

$$P(10,3) = \frac{10!}{7!}$$

Sit 6 people along a table



$$P(6,6) = \frac{6!}{0!} = \frac{6!}{1} = 6! = 720$$



pick 5 cards in order out of 52

$$P(52, 5) = \frac{52!}{47!}$$

How many ways can we arrange the letters A, B, C, D, E, F, G so that ABC are in that order?

ABC, D, E, F, G

$$P(5,5) = \frac{5!}{0!} = 120$$

An *r-combination* is an *unordered* arrangement of r objects.
(out of n)

$C(n,r)$

Ex: choose 3 people from a group of 4.

---

if order matters: ABC, ABD, ACD, BCD

BAC, BAD, CAD, CBD

CBA, DBA, DCA, DCB,

ACB ⋮

BCA ⋮

CAB

$$C(4,3) \times 3! = P(4,3)$$

$$C(n,r) = \frac{P(n,r)}{r!}$$

$$\boxed{C(n,r) = \frac{n!}{(n-r)!\,r!}} = \binom{n}{r}$$

"n choose r"

$$\binom{52}{5} = \frac{52!}{47!\,5!}$$

---

10 people, pick 3 $\binom{10}{3} = \frac{10!}{7!\,3!}$

$$\binom{n}{r} = \binom{n}{n-r}$$

10 people, pick 7 $\binom{10}{7} = \frac{10!}{3!\,7!}$

How many 8-bit strings contain exactly 5 1's?

_ 1 1 1 _ 1 1 _

$\binom{8}{5}$ $\binom{8}{3}$

How many 8-bit strings contain exactly r 1's? $\binom{8}{r}$

$$\binom{8}{0}+\binom{8}{1}+\binom{8}{2}+\binom{8}{3}+\binom{8}{4}+\binom{8}{5}+\binom{8}{6}+\binom{8}{7}+\binom{8}{8}$$

$$=2^8$$

$$\sum_{r=0}^{n}\binom{n}{r}=2^n$$

$$\sum_{r=0}^{n}\frac{n!}{n!(n-r)!}=2^n$$

Binomial Coefficients

$$(x+1)^0 = 1$$

$$(x+1)^1 = x+1$$

$$(x+1)^2 = x^2+2x+1$$

$$(x+1)^3 = x^3+3x^2+3x+1$$

$$(x+1)^4 = x^4+4x^3+6x^2+4x+1$$

$$(x+1)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} + \dots$$

$$+ \binom{n}{2}x^2 + \binom{n}{1}x + \binom{n}{0}$$

Binomial Coefficients

$$(x+1)^0 = 1$$

$$(x+1)^1 = x+1$$

$$(x+1)^2 = x^2+2x+1$$

$$(x+1)^3 = x^3+3x^2+3x+1$$

$$(x+1)^4 = x^4+4x^3+6x^2+4x+1$$

$$(x+1)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} + \cdots$$

$$+ \binom{n}{2}x^2 + \binom{n}{1}x + \binom{n}{0}$$