(d) $p \to q, q \to r, \neg(p \wedge r), p \vee r \Rightarrow r$.

(e) $\neg q, p \to q, p \vee t \Rightarrow t$

4. Give direct and indirect proofs of:

(a) $p \to q, \neg r \to \neg q, \neg r \Rightarrow \neg p$.

(b) $p \to \neg q, \neg r \to q, p \Rightarrow r$.

(c) $a \vee b, c \wedge d, a \to \neg c \Rightarrow b$.

5. Are the following arguments valid? If they are valid, construct formal proofs; if they aren't valid, explain why not.

(a) If wages increase, then there will be inflation. The cost of living will not increase if there is no inflation. Wages will increase. Therefore, the cost of living will increase.

(b) If the races are fixed or the casinos are crooked, then the tourist trade will decline. If the tourist trade decreases, then the police will be happy. The police force is never happy. Therefore, the races are not fixed.

6. Determine the validity of the following argument: For students to do well

~~studs hard → do well~~

do well → studs hard

2 (a) $\sum_{k=1}^{3} k^{n}$ for $n = 1, 2, 3, 4$

$n=1$: $\sum_{k=1}^{3} k^{1} = 1+2+3=6$

$n=2$: $\sum_{k=1}^{3} k^{2} = 1^{2}+2^{2}+3^{2} = 1+4+9=14$

(b) $\sum_{i=1}^{5} 20 = 20+20\ \ 20+20+20 = 100$

$$\sum_{k=-3}^{3} k = -3 + -2 + -1 + 0 + 1 + 2 + 3 = 0$$

$$\sum_{i=1}^{3} (a_i + b_i) = a_1 + b_1 + a_2 + b_2 + a_3 + b_3$$

$$= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)$$

$$= \sum_{i=1}^{3} a_i + \sum_{i=1}^{3} b_i$$

11. b is even   prov. shown, 10

※ 5, 11, a/b is in simplest form

$\therefore \sqrt{2} \neq \frac{a}{b}$

QED

$$\prod_{k=0}^{3} 2^k = 2^0 \cdot 2^1 \cdot 2^2 \cdot 2^3 = 1 \cdot 2 \cdot 4 \cdot 8 = 64$$

$$\prod_{k=1}^{100} \frac{k}{k+1} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdots \frac{\cancel{99}}{\cancel{100}} \cdot \frac{\cancel{100}}{101} = \frac{1}{101}$$

$F \to T$

$X = (p \wedge \neg q) \vee (r \wedge p)$

find a prop C such that $C \to X$

$(r \wedge p)$

$(p \wedge \neg q)$

F

$X \to p$

p

$$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$$

$$\neg\neg(p \vee q) \equiv p \vee q \equiv \neg((\neg p) \wedge (\neg q))$$

$$X = (p \wedge \neg q) \vee (p \wedge R)$$

P Q R	$p \wedge \neg q$	$p \wedge R$	X
0 0 0	0	0	0
0 0 1	0	0	0
0 1 0	0	0	0
0 1 1	0	0	0
1 0 0	1	0	1
1 0 1	1	1	1
1 1 0	0	0	0
1 1 1	0	1	1

Prove if a is even and b is odd then ~~$3a+5b$~~ $3a+5b$ is odd.

Proof:

1.	a is even	Hypothesis
2.	b is odd	Hypothesis
3.	$\exists k \in \mathbb{Z}: a = 2k$	Def of even, 1
4.	$\exists j \in \mathbb{Z}: b = 2j+1$	Def of odd, 2
5.	$3a+5b = 3 \cdot 2k + 5(2j+1)$	
	$= 6k + 10j + 5$	
	$= 2(3k+5j) + 5$	alg, subs, 3, 4
	$= 2(3k+5j) + 4 + 1$	
	$= 2(3k+5j) + 2 \cdot 2 + 1$	
	$= 2(\underline{3k+5j+2}) + 1$	
6.	$3k+5j+2 \in \mathbb{Z}$	Closure of ints, 3 ~~4~~
7.	$3a+5b$ is odd	Def of odd, 5, 6

QED

IGNORE

Want to show

$3a+5b$ is odd

$3a+5b = 2 \cdot L + 1$

Prove: if $n = ab$, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$

Proof:

1.	$\neg(a \le \sqrt{n}$ or $b \le \sqrt{n})$	Neg. of concl.
1.5	$n = ab$ ~~Hyp~~	Hyp
2.	$\neg(a \le \sqrt{n}) \land \neg(b \le \sqrt{n})$	De Morgan, 1
3.	$(a > \sqrt{n}) \land (b > \sqrt{n})$	~
4.	$ab > \sqrt{n} \cdot \sqrt{n}$	~
5.	$ab > n$	

⨳ 5, 1.5

$\therefore$ $a \le \sqrt{n}$ or $b \le \sqrt{n}$

QED

NUMBER THEORY

Divisibility

Def: An integer a is divisible by an integer b if

$$\left(\frac{a}{b} \in \mathbb{Z}\right) \quad \exists k \in \mathbb{Z} : a = b \cdot k$$

"b divides a" "b is a factor of a"

"a is a multiple of b"

Notation: $\underbrace{b \mid a}$ This is a proposition

"b divides a"

$b \nmid a$ means "b does not divide a"

Theorems

if $b|a$ and $b|c$ then $b|(a+c)$

proof:

1. $b|a$ Hyp
2. $b|c$ Hyp

~~3. $\exists k \in \mathbb{Z}: b = a \cdot k$ Def, 1~~

~~4. $\exists j \in \mathbb{Z}: b = c \cdot j$ Def, 2~~

No!

3. $\exists k \in \mathbb{Z}: a = b \cdot k$ Def, 1
4. $\exists j \in \mathbb{Z}: c = b \cdot j$ Def, 2
5. $a+c = b \cdot k + b \cdot j = b(k+j)$ alg, 3, 4
6. $k+j \in \mathbb{Z}$
7. $b|(a+c)$

QED

IGNORE

WTS $b|(a+c)$

WTS $a+c = b(\text{int})$

Thms: if $a \mid b$ and $a \mid c$ then $a \mid (b+c)$	$a=3,\ b=12,\ c=15$ $3 \mid (12+15)$
if $a \mid b$ then $a \mid bx$ $\forall x \in \mathbb{Z}$	$a=3,\ b=12$ $3 \mid 24$ $3 \mid 36$ $3 \mid 1200$
if $a \mid b$ and $b \mid c$ then $a \mid c$	$a=3$ $b=12$ $c=48$ $12 = 3 \cdot 4$

If $a \mid b$ and $a \mid c$ then

$a \mid mb + nc$ $\forall m,n \in \mathbb{Z}$

$48 = \underbrace{12}\cdot 4$

$(3 \cdot 4) \cdot 4$

THEOREM ("Division Algorithm")

Let $a \in \mathbb{Z}$, $b \in \mathbb{P}$ (positive int)

If $\frac{a}{b}$ is not an integer

$\frac{a}{b}$ has a remainder $< b$

and ≥ 1

In general, $\frac{a}{b}$ has a remainder r, $r \geq 0$, $r < b$

$\frac{13}{3} = 4$ Remainder 1

$\frac{13}{4}$

$13 = 3 \cdot 4 + 1$

$13 = 2 \cdot 4 + 5$

$13 = 1 \cdot 4 + 9$

$13 = 0 \cdot 4 + 13$

$13 = (-1) \cdot 4 + 17$

$13 = 4 \cdot 4 + (-3)$

13 % 4