Prove $1+3+\cdots+(2n-1)=n^2$ $\quad (n \geq 1)$

Let $P(n)$ be the prop: $1+3+5+\cdots+(2n-1)=n^2$ $\quad (n \geq 1)$

Prove $P(n)\ \forall n \geq 1$

Base Case: Prove $P(1)$

$P(1)$ says $1 = 1^2$ True

$\therefore P(1)$ is true

Inductive Step: Prove $P(n) \to P(n+1)$ $\quad (n \geq 1)$

1. $P(n)$ — Hypothesis
2. $1+3+5+\cdots+(2n-1)=n^2$ — Def of P
3. $1+3+5+\cdots+(2n-1)+\underline{(2n+1)} = n^2+\underline{(2n+1)}$ — Adding $(2n+1)$ to both sides of 2.
4. $1+3+5+\cdots+(2(n+1)-1) = (n+1)^2$ — Algebra

$\therefore P(n+1)$ — Def of P

$\therefore P(n) \to P(n+1)$

$\therefore P(n)\ \forall n \geq 1$ — Math. Ind.

Prove $1^3+2^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ $\forall n\geq 1$

Let $P(n)$ be the prop: $1^3+2^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

Prove $P(n)\ \forall n\geq 1$. Base Case: $P(1)$ says $1^3=\dfrac{1^2(1+1)^2}{4}=\dfrac{1^2\cdot 2^2}{4}=1$ ✓ $P(1)$ is true.

Induction Step: Prove $P(n)\to P(n+1)$

1. $P(n)$ Hyp.

2. $1^3+2^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ Def of P

3. $1^3+2^3+\cdots+n^3+(n+1)^3=\dfrac{n^2(n+1)^2}{4}+(n+1)^3$ add $(n+1)^3$ to both sides of 2

$$=\frac{n^2(n+1)^2+4(n+1)^3}{4}$$

$$=\frac{n^2(n^2+2n+1)+4(n^3+3n^2+3n+1)}{4}$$

$$=\frac{n^4+2n^3+n^2+4n^3+12n^2+12n+4}{4}$$

$$\frac{(n+1)^2(n+2)^2}{4}$$

$\therefore P(n+1)$

$P(n)\to P(n+1)$

$\therefore P(n)\ \forall n\geq 1$ QED

Math Ind.

$$= \frac{n^4 + 2n^3 + n^2 + 4n^3 + 12n^2 + 12n + 4}{4}$$

$$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

$$\frac{(n+1)^2 (n+2)^2}{4} = \frac{(n^2+2n+1)(n^2+4n+4)}{4} =$$

$$\frac{n^4 + 4n^3 + 4n^2 + 2n^3 + 8n^2 + 8n + n^2 + 4n + 4}{4} =$$

$$\frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

$$\frac{n^2(n+1)^2+4(n+1)^3}{4} =$$

$$\frac{n^2(n+1)^2+4(n+1)(n+1)^2}{4} =$$

$$\frac{\left(n^2+4(n+1)\right)(n+1)^2}{4}$$

$$\frac{n^2(n+1)^2 + 4(n+1)^3}{4} =$$

$$\frac{n^2(n+1)^2 + 4(n+1)(n+1)^2}{4} =$$

$$\frac{\left(n^2 + 4(n+1)\right)(n+1)^2}{4}$$

Prove $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n! = (n+1)! - 1$ $(n \geq 1)$

Let $P(n)$ be the prop "$1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1$"

Prove $P(n)$ $\forall n \geq 1$. Base case:

$P(1)$ says $1\cdot 1! = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$ ✓ $P(1)$ is true

Inductive Step: Prove $P(n) \Rightarrow P(n+1)$

1. $P(n)$ Hyp.

2. $1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1$ Def. of P

3. $1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)!$

$= \cancel{(n+1)!\,(\ldots)}$ $= (n+1)! + (n+1)!(n+1) - 1$

$= (n+1)!\,(1 + n+1) - 1$

$= (n+1)!\,(n+2) - 1$

$= (n+2)! - 1$

4. $P(n+1)$

Prove $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n! = (n+1)! - 1$ $(n \geq 1)$

Let $P(n)$ be the prop "$1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1$"

Prove $P(n)$ $\forall n \geq 1$. Base case:

$P(1)$ says $1\cdot 1! = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$ ✓ $P(1)$ is true

Inductive Step: Prove $P(n) \Rightarrow P(n+1)$

1. $P(n)$ Hyp.

2. $1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1$ Defn of P

3. $1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)!$

$= (n+1)!\,(\cancel{-1+n+1}) = (n+1)! + (n+1)!(n+1) - 1$

$= (n+1)!\,(1 + n+1) - 1$

$= (n+1)!\,(n+2) - 1$

$= (n+2)! - 1$

$(n+1)! = (n+1)(n!)$

4. $P(n+1)$

Prove $n(n+1)(n+2)$ is divisible by 3 $\forall n \geq 1$

Let $P(n)$ be the prop $3 \mid n(n+1)(n+2)$. Base case: $P(1)$ says $3 \mid 1\cdot2\cdot3$ ✓ $P(1)$ is true

Prove $P(n) \to P(n+1)$

1. $P(n)$ Hyp.

2. $3 \mid n(n+1)(n+2)$ Def of P

3. $\exists k \in \mathbb{Z}: n(n+1)(n+2) = 3k$ Def of "|"

4. $(n+3)[(n+1)(n+2)] = n[(n+1)(n+2)] + 3[(n+1)(n+2)]$

$$= 3k + 3[(n+1)(n+2)]$$

$$= 3(k + (n+1)(n+2))$$

Prove $n(n+1)(n+2)$ is divisible by 3 $\quad \forall n \geq 1$

Let $P(n)$ be the prop $3 \mid n(n+1)(n+2)$. Base case: $P(1)$ says $3 \mid 1 \cdot 2 \cdot 3$ ✓ $P(1)$ is true

Prove $P(n) \to P(n+1)$

1. $P(n)$ $\quad$ Hyp.
2. $3 \mid n(n+1)(n+2)$ $\quad$ Def of P
3. $\exists k \in \mathbb{Z}: n(n+1)(n+2) = 3k$ $\quad$ Def of "|"
4. $(n+3)[(n+1)(n+2)] = n[(n+1)(n+2)] + 3[(n+1)(n+2)]$

$$= 3k + 3[(n+1)(n+2)]$$

$$= 3(k + (n+1)(n+2))$$

Think (mod 3) $\qquad$ non-inductive proof

either $n \equiv 0 \quad n(n+1)(n+2) \equiv 0 \quad \therefore 3 \mid n(n+1)(n+2)$

$n \equiv 1 \quad n+1 \equiv 2 \quad n+2 \equiv 3 \equiv 0 \quad \therefore n(n+1)(n+2) \equiv 1 \cdot 2 \cdot 0 \equiv 0$

$n \equiv 2 \quad n+1 \equiv 0 \quad n+2 \equiv 1 \qquad \sim \equiv 2 \cdot 0 \cdot 1 \equiv 0$

that could be taken from $(0,0)$ to $(6,6)$, which is a path taken by traveling along grid lines going only to the right and up. How many different lattice paths are there of this type? Generalize to the case of lattice paths from $(0,0)$ to (m,n) for any nonnegative integers m and n.

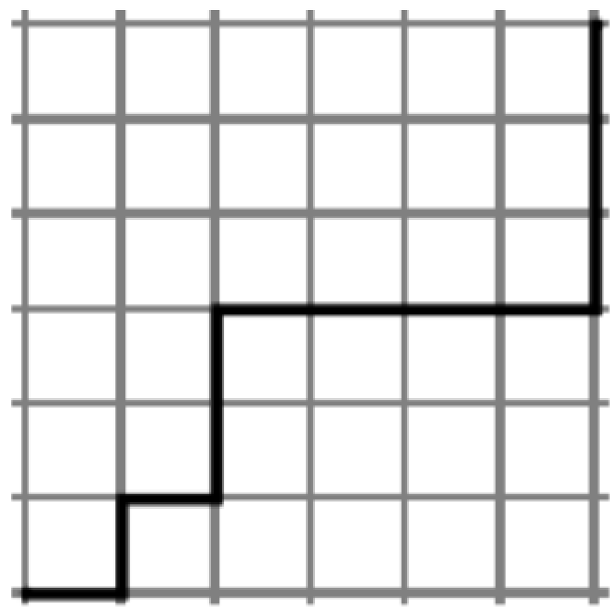
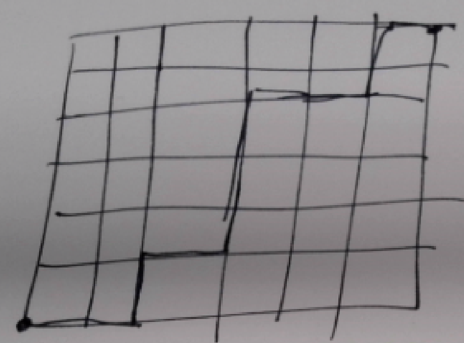


Figure 2.4.12 A lattice path

Hint. Think of each path as a sequence of instructions to go right (R) and up (U).



$$\binom{12}{6}$$

$$C(12,6) = \frac{12!}{6!\,6!}$$

R, U

RRU RUUUURRUUR

RRRRRR UUUUUU

6 R's, 6 U's

UUUUUU RRRRRR

$$\underbrace{\_\ \_\ \underline{R}\ \underline{R}\ \_\ \underline{R}\ \_\ \underline{R}\ \_\ \underline{R}\ \underline{R}\ \_}_{12}$$

How many ways can I place 6 R's in 12 spots?

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$n+1 = n+1$$

$$(n+1)^2 = n^2+2n+1$$

$$(n+1)^3 = n^3+3n^2+3n+1$$

$$(n+1)^4 = n^4+4n^3+6n^2+4n+1$$

$$(n+1)^5 = n^5+5n^4+10n^3+10n^2+5n+1$$

$$\binom{5}{0}n^5+\binom{5}{1}n^4+\binom{5}{2}n^3+\binom{5}{3}n^2+\binom{5}{4}n \qquad +1\binom{5}{5}$$

$$\boxed{(a+b)^n = \sum_{i=0}^{n}\binom{n}{i}a^i b^{n-i}}$$

Pascal's Triangle

1   1

1 1   2

1 2 1   4

1 3 3 1   8

1 4 6 4 1   16

1 5 10 10 5 1

1 6 15 20 15 6 1

$\downarrow r$

$\rightarrow c$

$\binom{7}{0}$ $\binom{7}{1}$ $\binom{7}{2}$ $\binom{7}{3}$ $\binom{7}{4}$ $\binom{7}{5}$ $\binom{7}{6}$ $\binom{7}{7}$

$\binom{n}{c}$

$$\sum_{i=1}^{n} \binom{n}{i} = 2^n$$

$11^0 = 1$

$11^1 = 11$

$11^2 = 121$

$11^3 = 1331$

$$(1+1)^n = \sum_{i=0}^{n} \binom{n}{i} \underbrace{1^i \, 1^{n-i}}_{1}$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

1
2
4
8
16

$\downarrow r$

$\rightarrow c$

$\binom{7}{0}\binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$

$\binom{n}{c}$

$$\sum_{i=1}^{n}\binom{n}{i} = 2^n$$

$11^0 = 1$

$11^1 = 11$

$11^2 = 121$

$11^3 = 1331$

$$(1+1)^n = \sum_{i=0}^{n}\binom{n}{i}\underbrace{1^i\,1^{n-i}}_{1}$$

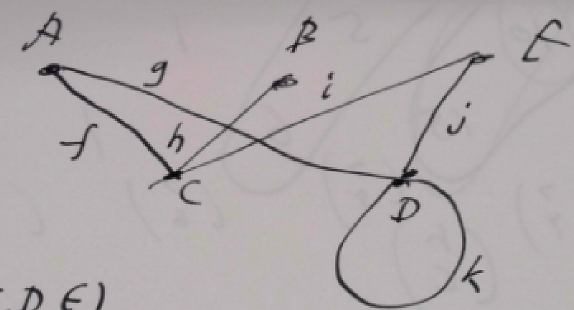
Graph Theory

A graph $G(V,E)$ is a collection of vertices (V) & edges (E)

$V \neq \emptyset$. An edge has 2 $\underbrace{\text{endpoints}}_{\in V}$ (could be same)

an edge "connects" its endpoints

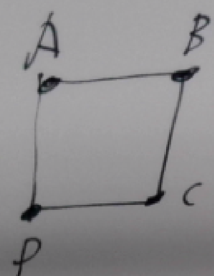
Ex:

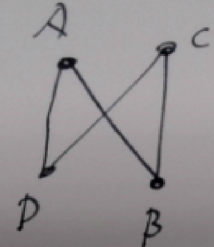


$V = \{A, B, C, D, E\}$

$E = \{f, g, h, i, j, k\}$

a "simple graph" has at most one edge for each pair of vertices.





friends