$\{x : x = x+1\}$ $\quad \{\}$ $\quad \emptyset$ $\quad \{1, 2, 3\}$

$A \times B$ $\quad \{3, 2, 1\}$

$\{1, 2, 3\}$

$A - B$

$\{x : x \in \{1, 2, 3\}\}$ $\quad \{1, 2, 3\} - \{3, 4, 5\} =$

$\{1, 2, 3\} \setminus \{3, 4, 5\} = \{1, 2\}$

$\{x \in \mathbb{Z} : x \geq 1 \text{ and } x \leq 3\}$

$1 \leq x \leq 3$

$\{x \in \mathbb{Z} : x = -x\}$

$\{x \in \mathbb{R} : x^2 < 10\}$

$\{0\}$ $\quad \{x : x = 0\}$

$\{x \in \mathbb{Q} : x = \frac{m}{n}, n \neq 0\}$

$\{x : x^2 = 9\}$ $\quad \cancel{\{x^2 = 9 : x\}}$

3.3.5 #2)

(a) $X = (p \wedge \neg q) \vee (r \wedge p)$

(b) find a prop B such that $B \equiv X$

(c) find a prop C such that $C \to X$

(d) find a prop D such that $X \to D$

Example: Let X be the prop: $P \wedge Q$

$Q \wedge P$

(c) $P \wedge Q \wedge R \to P \wedge Q$ $\quad\quad$ $P \to P \wedge Q$ (false)

(d) $P \wedge Q \to P$

$P \to Q$

$Q \to P$ converse

$\neg P \to \neg Q$ inverse

$\neg Q \to \neg P$ contrapositive

IDENTITIES

$P \wedge T \equiv P$ $\quad$ $P \vee F \equiv P$ $\quad$ "identity"

$P \wedge F \equiv F$ $\quad$ $P \vee T \equiv T$ $\quad$ "domination"

$P \wedge P \equiv P$ $\quad$ $P \vee P \equiv P$ $\quad$ "idempotence"

$\neg(\neg P) \equiv P$ $\quad$ "double negation"

$P \wedge Q \equiv Q \wedge P$ $\quad$ $P \vee Q \equiv Q \vee P$

$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ etc.

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ "distribution" the logical variables. For example, if p is "John owns a pet store" and q is "John likes pets," the detachment law should make sense.

Table 3.4.3 Basic Logical Laws - Equivalences

Commutative Laws	
$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
Associative Laws	
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
Distributive Laws	
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Identity Laws	
$p \vee 0 \Leftrightarrow p$	$p \wedge 1 \Leftrightarrow p$
Negation Laws	
$p \wedge \neg p \Leftrightarrow 0$	$p \vee \neg p \Leftrightarrow 1$
Idempotent Laws	
$p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$
Null Laws	
$p \wedge 0 \Leftrightarrow 0$	$p \vee 1 \Leftrightarrow 1$
Absorption Laws	
$p \wedge (p \vee q) \Leftrightarrow p$	$p \vee (p \wedge q) \Leftrightarrow p$
DeMorgan's Laws	
$\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$	$\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$$

$$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$

DeMorgan's Theorem

Example: Prove $\neg(P \vee (\neg P \wedge Q)) \equiv (\neg P) \wedge (\neg Q)$

$$\neg(P \vee (\neg P \wedge Q)) \equiv$$

$$\neg(P \vee Q) \equiv (\neg P) \overset{\wedge}{\cancel{\vee}} (\neg Q)$$

$$\neg(\square \vee \triangle) \equiv (\neg \square) \wedge (\neg \triangle)$$

$$\square = P \qquad \triangle = (\neg P \wedge Q)$$

$$\neg(\square \vee \triangle) \equiv (\neg \square) \wedge (\neg \triangle)$$

$$\equiv (\neg P) \wedge (\neg(\neg P \wedge Q))$$

$$\left.\begin{array}{l} \neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q) \\ \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q) \end{array}\right\} \text{DeMorgan's Th}$$

$$\left.\begin{array}{l} \neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q) \\ \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q) \end{array}\right\} \text{DeMorgan's Theorem}$$

Example: Prove $\neg(P \vee (\neg P \wedge Q)) \equiv (\neg P) \wedge (\neg Q)$

$$\begin{aligned}
\neg(P \vee (\neg P \wedge Q)) &\equiv \neg P \wedge \neg(\neg P \wedge Q) && \text{DeMorgan} \\
&\equiv \neg P \wedge (\neg\neg P \vee \neg Q) && \text{De Morgan} \\
&\equiv \neg P \wedge (P \vee \neg Q) && \text{Double Negation} \\
&\equiv (\neg P \wedge P) \vee (\neg P \wedge \neg Q) && \text{Distribution} \\
&\equiv F \vee (\neg P \wedge \neg Q) && \text{Negation law} \\
&\equiv (\neg P \wedge \neg Q) && \text{Identity law}
\end{aligned}$$

Q.E.D.

$$\neg(p \lor q) \equiv (\neg p) \overset{\land}{\cancel{\lor}} (\neg q)$$

$$\neg(\square \lor \triangle) \equiv (\neg \square) \land (\neg \triangle)$$

$$\square = p \qquad \triangle = (\neg p \land q)$$

$$\neg(\square \lor \triangle) \equiv (\neg \square) \land (\neg \triangle)$$

$$\equiv (\neg p) \land (\neg(\neg p \land q))$$

QUANTIFIERS

$\forall x$ "universal quantifier"

"for all x"

example $\underbrace{\forall x \ (x+1 > x)}_{\text{this is a proposition}}$

"for all x, $x+1$ is greater than x"

$\forall x \in \mathbb{Z} \ (x+1 > x)$ (T)

$\forall x \in \mathbb{Z} \ (x^2 \geq x)$ (T)

$\forall x \in \mathbb{R} \ (x^2 \geq x)$ (F: $(.1^2 = .01)$

$\exists x$ "existential quantifier"

"There exists"

example: $\exists x \in \mathbb{Z} : x^2 = x$ (T)

such that

$\underbrace{\exists x \in \mathbb{Z} : x = x+1}_{\text{This is a proposition}}$ (F)

A <u>family of propositions</u> $P(1), P(2), P(3), P(4)$

Ex: Let $P(i)$ be the proposition "$i^2 > i$"
$(i \in \mathbb{Z})$

$P(2)$ is the prop "$2^2 > 2$"

Let $P(i)$ be the prop "$i^2 > i$" $\quad i \in \mathbb{Z}$

$\forall i \in \mathbb{Z}\,(P(i))$ F

$\mathbb{N} = \{0, 1, 2, 3 \ldots\}$

$\exists i \in \mathbb{Z} : P(i)$ T

$\neg\left(\forall x\,(P(x))\right) \equiv \exists x : \neg P(x)$

$\neg\,(\exists x : P(x)) \equiv \forall x\,(\neg P(x))$

$\forall x \in \mathbb{N}\,(P(x)) \equiv P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \ldots$

$\exists x \in \mathbb{N} : P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \vee \ldots$

$$\forall x\,(P(x) \wedge Q(x)) \equiv \left(\forall x(P(x))\right) \wedge \left(\forall x(Q(x))\right)$$

$$\forall x\,(P(x) \vee Q(x)) \not\equiv \left(\forall x(P(x))\right) \vee \left(\forall x(Q(x))\right)$$

Let $P(x)$ be the prop "x is positive"

Let $Q(x)$ be the prop "$x \leq 0$"

T $\qquad\qquad$ $F \vee F$

F

$\forall x\ \exists y : x+y=0$ $\underline{T}$

for all x, there exists a y such that $x+y=0$.

$\exists y : \forall x\ (x+y=0)$ $\underline{F}$

There exists ~~an~~ a y such that $\forall x\ (x+y=0)$

always false

$\exists y : \forall x\ (xy=0)$ $T\ (y=0)$

$\exists x : \exists y : x=y$ T

$\forall x\ (\forall y\ (x=y))$ F

RULES OF INFERENCE

If it's sunny then I go for a run.

It's sunny.

I go for a run.

---

if this

and this

are true,

then so is this

$$\begin{array}{l} P \to Q \\ P \\ \hline Q \end{array}$$

Given that $P \to Q$ is true and P is true, we can conclude Q is true

$((P \to Q) \wedge P) \to Q \qquad (P \to Q) \wedge P \Rightarrow Q$

$$\begin{array}{l} P \to Q \\ \underline{P \qquad\quad} \\ Q \end{array}$$

"Modus Ponens"

"Direct Reasoning"

"Detachment"

$$\begin{array}{l} P \to Q \\ \underline{\neg Q \qquad\quad} \\ \neg P \end{array}$$

"Indirect Reasoning"