Functions

in $\to \boxed{f} \to$ out (A, B are sets)

A function $f: A \to B$ is an assignment (or a mapping) from each element $a \in A$ to an element in B called $f(a)$

How to specify?

ex: $f(a) = a^2$, $f: \mathbb{Z} \to \mathbb{Z}$

So $f(1) = 1$ $\quad f(-1) = 1$

$f(2) = 4$ $\quad f(-2) = 4$

$f(3) = 9$

$f: \mathbb{Z} \to \mathbb{Z}$

$f(a) =$ sum of all digits in a

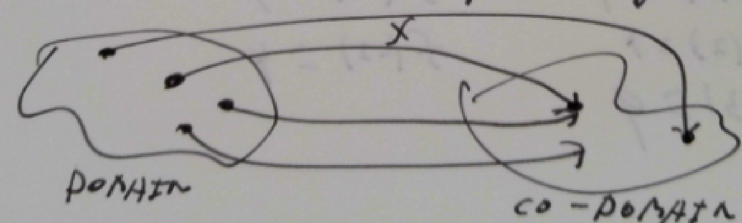
$f(123) = 6$

$f: A \to B$

~~the~~ Domain   Co-Domain

$f(a)$ is the "image" of a

"f of a" a is the "pre-image" of $f(a)$

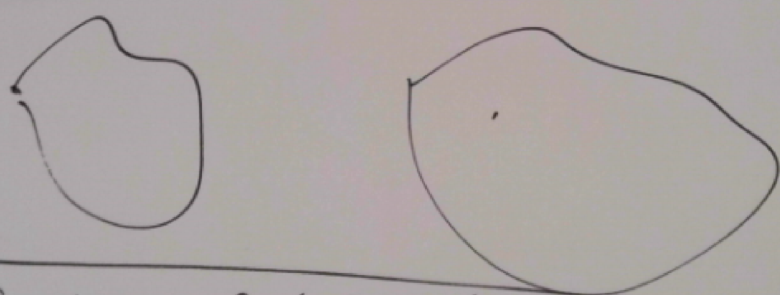


RANGE

ex: $f(x) = x^2$, $f: \mathbb{Z} \to \mathbb{Z}$

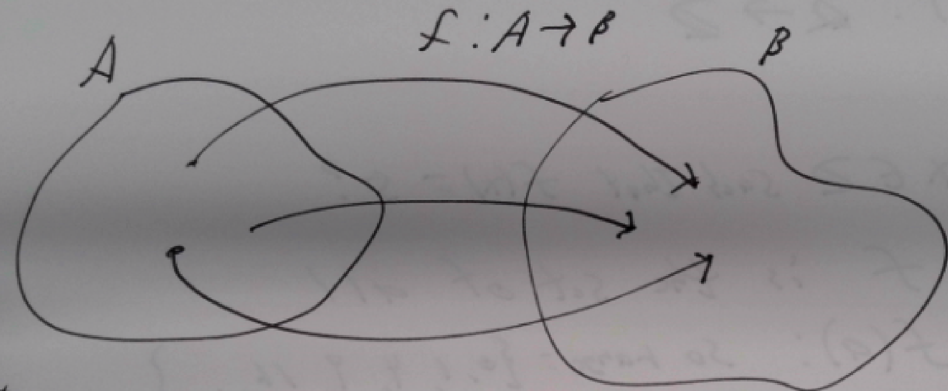
Is there some $x \in \mathbb{Z}$ such that $f(x) = 5$?

The range of f is the set of all $f(a)$: So range $= \{0, 1, 4, 9, 16, \ldots\}$



2 functions $f: A \to B$ & $g: C \to D$ are a equal if $A = C$, $B = D$, and $f(a) = g(a)$ for all $a \in A$

Characteristics of functions



I. If f takes different elements in A to different elements in B, f is <u>injective</u> "one-to-one"

II. If every element in B is mapped to from something, f is <u>surjective</u> "onto"

EXAMPLES

function	one-to-one injective?	onto Surjective?	one-to-one/onto Bijective
$f(x) = x^2$, $f: \mathbb{Z} \to \mathbb{Z}$	NO	NO	NO
$f(x) = x$, $f: \mathbb{Z} \to \mathbb{Z}$	YES	YES	YES
$f(x) = x$, $f: \mathbb{Z} \to \mathbb{R}$	YES	NO	NO
$f(x) = 2x$, $f: \mathbb{Z} \to \mathbb{Z}$	YES	NO	NO
$f(x) = 0$, $f: \mathbb{Z} \to \mathbb{Z}$	NO	NO	NO
$f(x) = \lvert x \rvert$, $f: \mathbb{Z} \to \mathbb{Z}$	NO	NO	NO
$f(x) = \tan x$, $f: ? \to \mathbb{Z}$			
$f(x)$ = Sum of digits, $f: \mathbb{Z} \to \mathbb{Z}$	NO		

"one-to-one correspondence"

SEQUENCES

ex: $1, 2, 3, 4, 5$ } finite

$3, 2, 1, 0$

$f(1) = 3$
$f(2) = 2$
$f(3) = 1$
$f(4) = 0$

$1, 2, 4, 8, 16, 32, \ldots$ } infinite

Def: a sequence of elements from a set S is a function from some subset $\mathbb{Z}$ to S.

ex: $f: \{1, 2, 3, 4, 5, \ldots\} \to \mathbb{Z}$

$a_1 = f(1) \qquad a_2 = f(2) \qquad a_3 = f(3)$ etc

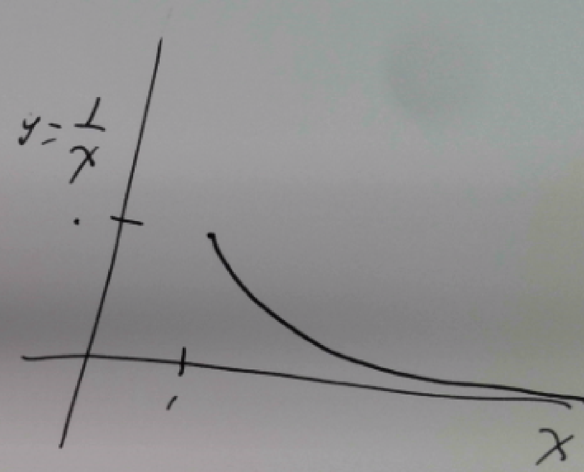
$a_1, a_2, a_3, \ldots$

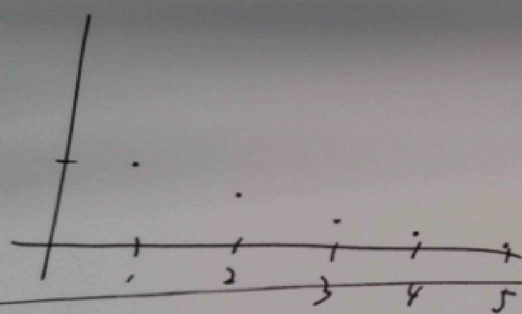
We can write $\{a_n\}$ for the sequence $a_1, a_2, a_3 \ldots$

Examples

$a_n = \frac{1}{n}, \quad n = 1, 2, 3, 4, \ldots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$





DEF: A recurrence relation for a sequence $a_1, a_2, a_3, \ldots$ is an expression for a_n in terms of $a_1, a_2, \ldots, a_{n-1}$

given such an expression the sequence $a_1, a_2, a_3, \ldots$ is called "a solution" of the recurrence relation

	initial condition
ex: $1, 2, 4, 8, 16, 32 \ldots$ $a_n = 2 \cdot a_{n-1}$	$a_1 = 1$
$0, 1, 3, 6, 10, 15, \ldots$ $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$ $a_n = a_{n-1} + n$ $a_6 = a_5 + 6 = 21$	$a_0 = 0$
$0, 1, 2, 3, 4, 5, \ldots$ $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$ $(a_n = n$ or$)$ $a_n = a_{n-1} + 1$	$a_0 = 0$
$1, 2, 6, 24, 120, 720, 5040, \ldots$ $a_n = n \cdot a_{n-1}$	$a_1 = 1$

$a_n = a_{n-1} + a_{n-2} \quad (n > 1)$

$a_1 = a_0 = 1$

$1, 1, 2, 3, \overbrace{5, 8, 13}^{65}, 21, 34, 55, 87, \ldots$

fibonacci

$64 \quad \frac{21}{13} \quad \frac{34}{21} \quad \frac{55}{34} \quad \ldots$

Interest

5% annual interest.

$P_0 = 100$

$P_1 = 105 = P_0 + 0.05 \cdot P_0 = 1.05 \cdot P_0$

$P_2 = P_1 + 0.05 P_1 = 1.05 P_1 = (1.05)(1.05) P_0 = (1.05)^2 \cdot P_0$

$P_3 = \cdots = (1.05)^3 P_0$

$\vdots$

$P_n = (1.05)^n \cdot P_0$

Summations

$$x_1 + x_2 + x_3 + x_4 + x_5$$

$\sum_{i=1}^{5} x_i$ means this

$\sum_{i=3}^{6} x_i$ means $x_3 + x_4 + x_5 + x_6$

index of summation

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sum=0
for i = 3 to 6
    sum = sum + x[i]
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$$\sum_{j=0}^{3} x_{j+3} = x_3 + x_4 + x_5 + x_6$$

$$\sum_{i=0}^{n} \left(f(i) + g(i) \right) = \left(\sum_{i=0}^{n} f(i) \right) + \left(\sum_{i=0}^{n} g(i) \right)$$

$$\sum_{i=0}^{n} c \cdot f(i) = c \cdot \sum_{i=0}^{n} f(i)$$

Double Summations

$$\sum_{i=1}^{3} \left[\sum_{j=2}^{4} (j \cdot i) \right] = \sum_{i=1}^{3} 9i = 9 \cdot 1 + 9 \cdot 2 \times 9 \cdot 3$$

$$= 9 + 18 + 27 = 54$$

$$\sum_{j=2}^{4} (j \cdot i) = 2 \cdot i + 3i + 4 \cdot i = 9i$$

$$1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 3\cdot 2 + 3\cdot 3 + 3\cdot 4$$

$$1\sum_{j=2}^{4} j + 2\sum_{j=2}^{4} j + 3\sum_{j=2}^{4} j = (1+2+3)(2+3+4)$$

$$\sum_{i=0}^{\infty} \sim \qquad\qquad \sum_{-\infty}^{\infty}$$

$$\sum_{i=0}^{\infty} 2^{-i} = 2^{0} + 2^{-1} + 2^{-2} + 2^{-3} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 3\cdot 2 + 3\cdot 3 + 3\cdot 4$$

$$1\sum_{j=2}^{4} j + 2\sum_{j=2}^{4} j + 3\sum_{j=2}^{4} j = (1+2+3)(2+3+4)$$

$$\sum_{i=0}^{\infty} \sim \qquad \sum_{-\infty}^{\infty}$$

$$\sum_{i=0}^{\infty} 2^{-i} = 2^{0} + 2^{-1} + 2^{-2} + 2^{-3} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$\prod_{i=1}^{n} f(i) = f(1) \times f(2) \times f(3) \times \cdots \times f(n)$$

Matrices

Rectangular array of numbers

$$\left.\underbrace{\begin{bmatrix} 2 & 4 & -6 \\ 3 & 18 & 2.5 \\ 9 & 11 & 17 \\ -1 & 0 & -1 \\ 5 & 15 & 14 \end{bmatrix}}_{\text{3 columns}}\right\}\text{5 rows}$$

~~3 rows~~

3 columns

5 x 3 matrix

$[a_{i,j}]$ row i column j

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix}$$

$$- \qquad \begin{bmatrix} -3 & 1 \\ 3 & 3 \end{bmatrix}$$

$$10 \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 40 & 30 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1\ 2)\cdot(4\ 1) = 1\cdot 4 + 2\cdot 1 = 6 & (1\ 2)\cdot(1\ 0) = 1\cdot 1 + 2\cdot 0 = 1 \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$ (Swap rows & columns)

If $M = M^T$, M is called symmetric

I = identity matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{A \cdot I = A = I \cdot A}$

Note: in general, $AB \neq BA$

$A \div B$ not really defined.

Sometimes given a matrix A,

we can find another matrix A^{-1} ("the inverse of A")

such that $A \cdot A^{-1} = I$

A^{-1} looks like $\frac{1}{A}$

BA^{-1} looks like $\frac{B}{A}$

$$AB = C$$

$$A^{-1}AB = A^{-1}C$$

$$IB = A^{-1}C$$

$$B = A^{-1}C$$