Signals & Systems - Chapter 5

1S. Use the Fourier transform analysis equation to calculate the Fourier transforms of:

a) $(\frac{1}{2})^{n-1}u[n-1]$ **b)** $(\frac{1}{2})^{|n-1|}$

Solution: For discrete-time Fourier transform, we have:

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{jw})e^{jwn}dw \qquad \textit{Synthesis equation}$$

$$X(e^{jw})=\sum_{n=-\infty}^{+\infty}x[n]e^{-jwn} \qquad \textit{Analysis equation}$$

a)

$$X(e^{jw})=\sum_{n=-\infty}^{+\infty}x[n]e^{-jwn} \qquad \textit{Analysis equation}$$

$$X(e^{jw})=\sum_{n=1}^{+\infty}(\frac{1}{2})^{n-1}e^{-jwn}=\sum_{n=0}^{+\infty}(\frac{1}{2})^{n}e^{-jw(n+1)}=e^{-jw}\sum_{n=0}^{+\infty}\left(\frac{1}{2}e^{-jw}\right)^{n}$$

Apply Infinite Sum

$$X(e^{jw})=e^{-jw}\frac{1}{1-(\frac{1}{2}e^{-jw})}$$

<u>Note</u>

We have Finite sum $\sum_{n=0}^{N}a^n=\frac{1+a^N}{1-a}$ *and if |a|<1 & N→∞ then we have infinite sum* $\sum_{n=0}^{\infty}a^n=\frac{1}{1-a}$

b)

$$X(e^{jw})=\sum_{n=-\infty}^{+\infty}x[n]e^{-jwn} \qquad \textit{Analysis equation}$$

$$X(e^{jw})=\sum_{n=-\infty}^{1}(\frac{1}{2})^{-(n-1)}e^{-jwn}+\sum_{n=2}^{\infty}(\frac{1}{2})^{(n-1)}e^{-jwn}=\sum_{n=0}^{\infty}(\frac{1}{2})^{n}e^{-jw(-n+1)}+\sum_{n=0}^{\infty}(\frac{1}{2})^{n}e^{-jw(n+1)}$$

$$X(e^{jw})=(e^{-jw})\sum_{n=0}^{+\infty}\left((\frac{1}{2})e^{jw}\right)^{n}+(e^{-jw})\sum_{n=0}^{+\infty}\left((\frac{1}{2})e^{-jw}\right)^{n}$$

Apply Finite Sum

$$X(e^{jw})=\frac{e^{-jw}}{1-\frac{1}{2}e^{+jw}}+\frac{e^{-jw}}{1-\frac{1}{2}e^{-jw}}$$

1U. Use the Fourier transform analysis equation to calculate the Fourier transforms of:

a) $(\frac{1}{2})^{n+1}u[n+1]$ **b)** $(\frac{1}{2})^{|n-5|}$

Solution:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad \textit{Synthesis equation}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn} \qquad \textit{Analysis equation}$$

2S. Use the Fourier transform synthesis equation to determine the inverse Fourier transform of:

a) $X_1(e^{jw}) = \sum_{k=-\infty}^{\infty}\{4\pi\delta(w-4\pi k) + 2\pi\delta(w-\frac{\pi}{2}-3\pi k) + 2\pi\delta(w+\frac{\pi}{2}-3\pi k)\}$

b) $X_2(e^{jw}) = \begin{cases} 2j & for \quad 0 < w \le \pi \\ -2j & for \quad -\pi < w \le 0 \end{cases}$

Solution:

a)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad \textit{Synthesis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty}\{4\pi\delta(w-4\pi k) + 2\pi\delta(w-\frac{\pi}{2}-3\pi k) + 2\pi\delta(w+\frac{\pi}{2}-3\pi k)\}e^{jwn}dw$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{4\pi\delta(w) + 2\pi\delta(w-\frac{\pi}{2}) + 2\pi\delta(w+\frac{\pi}{2})\}e^{jwn}dw$$

$$x[n] = 2e^{j0} + e^{j(\pi/2)n} + e^{-j(\pi/2)n} = 2 + 2\cos(\pi n/2)$$

b)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad \textit{Synthesis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{0}^{\pi} 2je^{jwn}dw - \frac{1}{2\pi}\int_{-\pi}^{0} 2je^{jwn}dw = (j/\pi)\left[-\frac{1-e^{+jn\pi}}{jn} + \frac{e^{-jn\pi}-1}{jn}\right]$$

$$x[n] = \frac{2}{n\pi}[-1 + Cos(n\pi)] = -(4/(n\pi))\sin^2(n\pi/2)$$

2U. Use the Fourier transform synthesis equation to determine the inverse Fourier transform of:

a) $X_1(e^{jw}) = \sum_{k=-\infty}^{\infty}\{2\pi\delta(w-2\pi k) + \pi\delta(w-\frac{\pi}{3}-2\pi k) + \pi\delta(w+\frac{\pi}{4}-5\pi k)\}$

b) $X_2(e^{jw}) = \begin{cases} 5j & for \quad 0 < w \le 2\pi/3 \\ -5j & for \quad -2\pi/3 < w \le 0 \\ 0 & Otherwise \end{cases}$

Solution:

3S. Given that x[n] has Fourier transform $X(e^{jw})$, express the Fourier transform of the following signals in terms of $X(e^{jw})$. You may use the Fourier transform properties table.

a) $x_1[n] = x[1 - n] + x[-1 - n]$

b) $x_2[n] = \dfrac{x^*[-n] + x[n]}{2}$

c) $x_3[n] = (n - 1)^2 x[n]$

Solution:

a)

$$Time\ \text{Re}versal: \quad x[-n] \xrightarrow{FT} X(e^{-jw})$$

$$Time\ Shift: \quad x[n-n_0] \xrightarrow{FT} e^{jwn_0} X(e^{jw})$$

$$Therefore$$

$$x[n] = x[1-n] + x[-1-n] \xrightarrow{FT} e^{jw} X(e^{-jw}) + e^{-jw} X(e^{-jw}) = 2X(e^{-jw}) \cos w$$

b)

$$Time\ \text{Re}versal: \quad x[-n] \xrightarrow{FT} X(e^{-jw})$$

$$Conjugation\ \text{Pr}operty: \quad x^*[n] \xrightarrow{FT} X^*(e^{-jw})$$

$$Therefore$$

$$x[n] = (1/2)\{x^*[-n] + x[n]\} \xrightarrow{FT} (1/2)\{X^*(e^{jw}) + X(e^{jw})\} = \text{Re}al\{X(e^{jw})\}$$

c)

$$Differentiation\ in\ Frequency: \quad nx[n] \xrightarrow{FT} j\frac{dX(e^{jw})}{dw}$$

$$Differentiation\ in\ Frequency\ Again: \quad n^2 x[n] \xrightarrow{FT} -\frac{d^2X(e^{jw})}{dw^2}$$

$$Therefore$$

$$x[n] = (n-1)^2 x[n] = (n^2 - 2n + 1)x[n] \xrightarrow{FT} -\frac{d^2X(e^{jw})}{dw^2} - 2j\frac{dX(e^{jw})}{dw} + X(e^{jw})$$

3U. Given that x[n] has Fourier transform $X(e^{jw})$, express the Fourier transform of the following signals in terms of $X(e^{jw})$. You may use the Fourier transform properties table.

a) $x_1[n] = x[1 + n] + x[1 - n]$

b) $x_2[n] = \dfrac{2x[-n] + 4x^*[n]}{3}$

c) $x_3[n] = (n + 3)^2 x[n-1]$

Solution:

4S. For each of the following Fourier transforms, use Fourier transform properties table to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

a) $X(e^{jw}) =$ $e^{-jw} \sum_{k=1}^{10} (\sin kw)$

b) $X(e^{jw}) =$ $j \sin(w) \cos(5w)$

c)

$$X(e^{jw}) = A(w) + e^{+jB(w)}$$

$$where \quad B(w) = -\frac{3w}{2} + \pi \quad and \quad A(w) = \begin{cases} 1 & for \quad 0 \le |w| \le \frac{\pi}{8} \\ 0 & for \quad \frac{\pi}{8} < |w| \le \pi \end{cases}$$

Solution:

<u>Relevant Properties (Source Properties Table)</u>

1) X(e^{jw}) real and even → x[n] real and even
2) X(e^{jw}) purely imaginary and Odd → x[n] real and odd
3) X(e^{jw}) pure imaginary → x[n] real
4) X(e^{jw}) pure real → x[n] real

a)

We have property #2 therefore the reverse is true also meaning

2') x_1[n] real and odd → $X_1(e^{jw})$ purely imaginary and Odd

We see that $X_1(e^{jw}) = \sum_{k=1}^{10} (\sin kw)$ is real and odd therefore Corresponding signal x_1[n] is purely imaginary and odd.

x[n]=x_1[n-1] to count for e^{-jw}

therefore x[n] is also purely imaginary but it is neither odd or even.

b)

Since X(e^{jw}) is purely imaginary and odd → x[n] is real and odd (Property #2)

c)

Since X(e^{jw}) = X*(e^{-jw}) → x[n] is real but neither odd or even.

4U. For each of the following Fourier transforms, use Fourier transform properties table to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

a) X(e^{jw}) = $e^{-2jw} \sum_{k=2}^{15} (\sin kw)$

b) X(e^{jw}) = $j \sin(3w) \cos(2w)$

c)

$$X(e^{jw}) = A(w) + e^{+jB(w)}$$

$$where \quad B(w) = +\frac{5w}{2} - \pi \quad and \quad A(w) = \begin{cases} 1 & for \quad 0 \le |w| \le \frac{\pi}{8} \\ 0 & for \quad \frac{\pi}{8} < |w| \le \pi \end{cases}$$

Solution:

5S. Compute the Fourier transform of each of the following signals:

a) x[n] = u[n – 2] – u[n – 6]

b) x[n] = $(1/3)^{|n|}$ u[–n – 2]

c) x[n] = (1/2) $^{|n|}$ cos{(π/8)(n-1)}
d) x[n] = sin(nπ/2) + cos(n)
e) x[n] = $(\frac{\sin(\pi n/5)}{\pi n})\cos(\frac{7\pi}{2}n)$

Solution:

a) We can rewrite the signal as

$$x[n] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$$

Based on transformation table & time shift property

$$x(e^{jw}) = e^{-j2w} + e^{-j3w} + e^{-j4w} + e^{-j5w}$$

b) Use the Fourier Transform Equation

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{-2} (\frac{1}{3})^{-n} e^{-jwn} = \sum_{n=2}^{\infty} (\frac{1}{3})^{n} e^{jwn} = \sum_{n=0}^{\infty}\left(\frac{1}{3}e^{jw}\right)^{n+2} = \frac{1}{9}e^{j2w}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{jw}\right)^{n}$$

$$\text{Finite sum } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{where} \quad a<1$$

$$X(e^{jw}) = \frac{e^{j2w}}{9}\frac{1}{1-\frac{1}{3}e^{jw}}$$

c) Use the Fourier Transform Equation

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} (\frac{1}{2})^{|n|}\cos(\frac{\pi(n-1)}{8})e^{-jwn} = \frac{1}{2}\sum_{n=-\infty}^{\infty} (\frac{1}{2})^{|n|}\left[e^{j(\frac{\pi(n-1)}{8})} + e^{-j(\frac{\pi(n-1)}{8})}\right]e^{-jwn}$$

$$X(e^{jw}) = \frac{1}{2}\sum_{n=-\infty}^{\infty} (\frac{1}{2})^{|n|}\left[e^{-j\frac{\pi}{8}}e^{j(\frac{\pi}{8}-w)n} + e^{j\frac{\pi}{8}}e^{j(-\frac{\pi}{8}-w)n}\right] =$$

$$X(e^{jw}) = \frac{e^{-j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left[\frac{1}{2}e^{j(-\frac{\pi}{8}-w)}\right]^n + \left[\frac{1}{2}e^{j(\frac{\pi}{8}-w)}\right]^n + \frac{e^{j\frac{\pi}{8}}}{2}\sum_{n=-\infty}^{0}\left[\frac{1}{2}e^{j(\frac{\pi}{8}+w)}\right]^n + \left[\frac{1}{2}e^{j(-\frac{\pi}{8}+w)}\right]^n$$

$$\text{Finite sum } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{where} \quad a<1$$

$$X(e^{jw}) = \frac{e^{-j\frac{\pi}{8}}}{2(1-\frac{1}{2}e^{j(-\frac{\pi}{8}-w)})} + \frac{e^{-j\frac{\pi}{8}}}{2(1-\frac{1}{2}e^{j(\frac{\pi}{8}-w)})} + \frac{e^{j\frac{\pi}{8}}}{2(1-\frac{1}{2}e^{j(\frac{\pi}{8}+w)})} + \frac{e^{j\frac{\pi}{8}}}{2(1-\frac{1}{2}e^{j(-\frac{\pi}{8}+w)})}$$

d) Use the Fourier Transform Table

$$X(e^{jw}) = \sin(\pi n/2) + \cos(n) = \frac{1}{2j}[e^{j\pi n/2} - e^{-j\pi n/2}] + \frac{1}{2}[e^{jn} + e^{-jn}]$$

$$\text{From table } \quad e^{jw_0 n} \xrightarrow{F.T.} 2\pi\sum_{l=-\infty}^{\infty}\delta(w - w_0 - 2\pi l)$$

$$X(e^{jw}) = \frac{2\pi}{2j}[\sum_{l=-\infty}^{\infty}\delta(w-\frac{\pi}{2}-2\pi l) - \sum_{l=-\infty}^{\infty}\delta(w+\frac{\pi}{2}-2\pi l)] + \frac{2\pi}{2}[\sum_{l=-\infty}^{\infty}\delta(w-1-2\pi l) + \sum_{l=-\infty}^{\infty}\delta(w+1-2\pi l)]$$

e)

$$x[n] = (\frac{\sin(\pi n/5)}{\pi n})\cos(\frac{7\pi}{2}n) = (\frac{\sin(\pi n/5)}{\pi n})(1/2)(e^{j\frac{7\pi}{2}n} + e^{-j\frac{7\pi}{2}n})$$

$$From\ FT\ Table\ \frac{\sin(\pi n/5)}{\pi n} \xrightarrow{FT} x[w] = 1 \quad for \quad 0 \le |w| \le \frac{\pi}{5}$$

$$0 \quad otherwise$$

$$x[w]\ periodic\ with\ period\ 2\pi$$

$$From\ FT\ \mathrm{Pr}operty\ Table\ e^{jw_0 n}x[n] \xrightarrow{FT} X(e^{j(w-w_0)})$$

$$Therefore\ \frac{\sin(\pi n/5)}{\pi n}\cos(\frac{7\pi}{2}n) \xrightarrow{FT}$$

$$x[w] = \{\frac{1}{2} \quad for \quad 0 \le |w - \frac{7\pi}{2}| \le \frac{\pi}{5}\} + \{\frac{1}{2} \quad for \quad 0 \le |w + \frac{7\pi}{2}| \le \frac{\pi}{5}\}$$

$$0 \quad otherwise$$

5U. Compute the Fourier transform of each of the following signals:

a) x[n] = u[n + 2] – u[n – 5]

b) x[n] = $(1/5)^{|n|}$ u[–n – 6]

c) x[n] = $(1/3)^{|n|}$ cos{(π/6)(n+3)}

d) x[n] = sin(nπ/3) + cos(n)

Solution:

6S. The following are the Fourier transforms of discrete-time signals. Determine the signal corresponding to each transform.

a)

$$X(e^{jw}) = 1 \quad for \quad \frac{\pi}{4} \le |w| \le \frac{3\pi}{4}$$

$$0 \quad for \quad \frac{3\pi}{4} \le |w| \le \pi \quad and \quad 0 \le |w| \le \frac{\pi}{4}$$

b) $X(e^{jw}) = e^{-jw/2} \quad for \quad -\pi \le w \le \pi$

c) $X(e^{jw}) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(w - \frac{\pi}{2}k)$

d) $X(e^{jw}) = \dfrac{1 - \frac{1}{3}e^{-jw}}{1 - \frac{2}{3}e^{-jw} + \frac{1}{9}e^{-2jw}}$

Solution:

a)

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{jw})e^{jwn}dw=\frac{1}{2\pi}\int_{\pi/4}^{3\pi/4}e^{jwn}dw+\frac{1}{2\pi}\int_{-3\pi/4}^{-\pi/4}e^{jwn}dw$$

$$x[n]=\frac{1}{2\pi(jn)}\{e^{jn(\frac{3\pi}{4})}-e^{jn(\frac{\pi}{4})}\}+\frac{1}{2\pi(jn)}\{e^{jn(\frac{-\pi}{4})}-e^{jn(\frac{-3\pi}{4})}\}$$

$$x[n]=\frac{1}{2\pi(jn)}\{2j\sin(n\frac{3\pi}{4})-2j\sin(n\frac{\pi}{4})\ \}=\frac{1}{\pi n}\{\sin(n\frac{3\pi}{4})-\sin(n\frac{\pi}{4})\}$$

b)

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{jw})e^{jwn}dw=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-jw/2}e^{jwn}dw=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j(n-\frac{1}{2})w}dw$$

$$x[n]=\frac{1}{2\pi j(n-\frac{1}{2})}(e^{j(n-\frac{1}{2})w}|_{-\pi}^{\pi}=\frac{1}{2\pi j(n-\frac{1}{2})}(e^{j(n-\frac{1}{2})\pi}-e^{-j(n-\frac{1}{2})\pi})=\frac{2j}{2\pi j(n-\frac{1}{2})}\sin(n\pi-\frac{\pi}{2})$$

$$x[n]=\frac{1}{\pi(n-\frac{1}{2})}\sin(n\pi-\frac{\pi}{2})$$

c) $X(e^{jw}) = \sum_{k=-\infty}^{\infty}(-1)^k\,\delta(w-\frac{\pi}{2}k)$ is periodic with fundamental period of 4 therefore fundamental frequency $w_0 = \frac{\pi}{2}$, also $a_k = (-1)^k$. Using these facts we could use the Fourier series synthesis equation to find x[n]

$$x[n]=\sum_{k=<N>}a_k e^{jk(\frac{2\pi}{N})n}=\sum_{k=0}^{3}a_k e^{jk(\frac{2\pi}{N})n}=1-e^{jn\pi/2}+e^{jn\pi}-e^{j3n\pi/2}$$

d) Rewrite the function by doing a long division

$$X(e^{jw})=\frac{1-\frac{1}{3}e^{-jw}}{1-\frac{2}{3}e^{-jw}+\frac{1}{9}e^{-2jw}}=\frac{1-\frac{1}{3}e^{-jw}}{\left(1-\frac{1}{3}e^{-jw}\right)^2}=\frac{1}{1-\frac{1}{3}e^{-jw}}$$

$$Given\Rightarrow\sum_{n=0}^{\infty}a^n=\frac{1}{1-a}\quad where\ a<1$$

$$X(e^{jw})=1+\frac{1}{3}e^{-jw}+\frac{1}{3^2}e^{-j2w}+\frac{1}{3^3}e^{-j3w}+\frac{1}{3^4}e^{-j4w}+\frac{1}{3^5}e^{-j5w}+\ldots$$

from F.T. table

$$From\ table\quad e^{-jwn_0}\xrightarrow{I.F.T.}\delta(n-n_0)$$

$$X[n]=\delta(n)+\frac{1}{3}\delta(n-1)+\frac{1}{3^2}\delta(n-2)+\frac{1}{3^3}\delta(n-3)+\frac{1}{3^4}\delta(n-4)+\frac{1}{3^5}\delta(n-5)+\ldots$$

6U. The following are the Fourier transforms of discrete-time signals. Determine the signal corresponding to each transform.

a) $X(e^{jw}) = e^{-jw/3}\quad for\quad -2\pi/3\le w\le 2\pi/3$

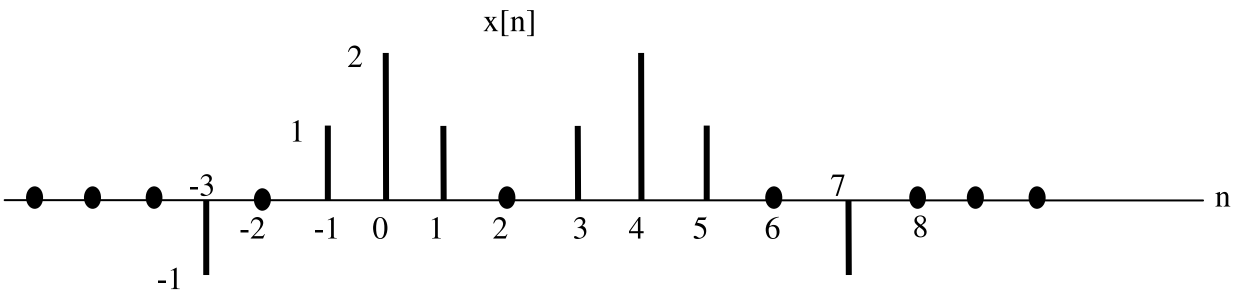
0 Otherwise

b) $X(e^{jw}) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(w - \frac{\pi}{3}k)$

c) $X(e^{jw}) = \dfrac{1 + \frac{1}{5}e^{-jw}}{-1 - \frac{2}{5}e^{-jw} - \frac{1}{25}e^{-2jw}}$

Solution:

7S. Let $X(e^{jw})$ denote the Fourier transform of the signal x[n] depicted in the following figure:



Perform the following calculations without explicitly evaluating $X(e^{jw})$:

a) **Evaluate $X(e^{j0})$**
b) **Find the phase of $X(e^{jw})$**
c) **Evaluate** $\int_{-\pi}^{\pi} X(e^{jw})dw$
d) **Find $X(e^{j\pi})$**
e) **Determine and sketch the signal whose Fourier transform is Real{$X(e^{jw})$}**
f) **Evaluate** $\int_{-\pi}^{\pi} |X(e^{jw})|^2 dw$
g) **Evaluate** $\int_{-\pi}^{\pi} |\frac{dX(e^{jw})}{dw}|^2 dw$

Solution:

a) write the Fourier Transform Equation

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n]e^{0} = -1 + 0 + 1 + 2 + 1 + 0 + 1 + 2 + 1 + 0 - 1 = 6$$

b) Note that the function y[n]=x[n+2] is an even signal, therefore $Y(e^{jw})$ is real and even. This implies that the phase of $Y(e^{jw})=0$.
Using the time shifting property of the Fourier transform we have $Y(e^{jw})=e^{j2w}X(e^{jw})$ therefore phase of $Y(e^{jw})$ is -2w.

c)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}\,dw \quad \textit{Inverse Fourier Transform}$$

$$2\pi x[0] = \int_{2\pi} X(e^{jw})dw$$

$$\textit{We have} \quad x[0] = 2 \Rightarrow \int_{2\pi} X(e^{jw})dw = 4\pi$$

d)

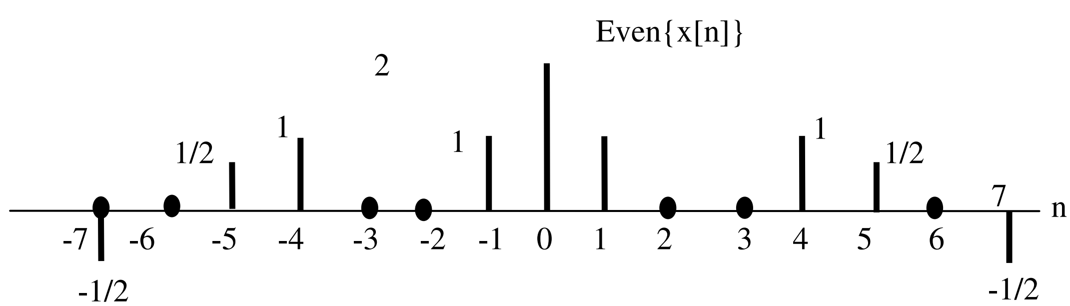
$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\pi n} = \sum_{n=-\infty}^{+\infty} x[n](-1)^n = +1+0-1+2-1+0-1+2-1+0+1 = 2$$

e) From property table we have the following

$$even\{x[n]\xrightarrow{FT}\mathrm{Re}\{X(e^{jw})\}$$

$$\textit{Therefore, the desired signal is } even\{x[n]\} = (x[n]+x[-n])/2 \textit{ shown below:}$$



f) From Parseval theorem we have

$$\frac{1}{2\pi}\int_{2\pi}|X(e^{jw})|^2\,dw = \sum_{n=-\infty}^{+\infty}|x[n]|^2 = 14 \Rightarrow \int_{2\pi}|X(e^{jw})|^2\,dw = 28\pi$$

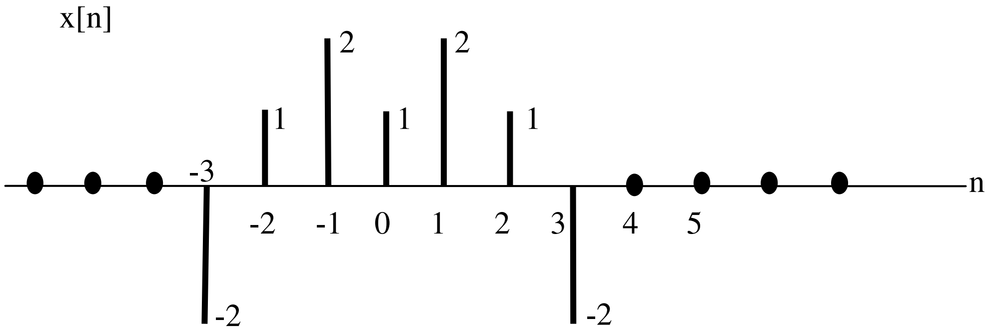
g) Using the differentiation in frequency property of Fourier transform

$$nx[n]\xrightarrow{FT} j\frac{dX(e^{jw})}{dw}$$

Use Parseval's therorem

$$\frac{1}{2\pi}\int_{2\pi}\left|j\frac{dX(e^{jw})}{dw}\right|^2 dw = \sum_{n=-\infty}^{+\infty}|nx[n]|^2 = 158 \Rightarrow \int_{2\pi}\left|j\frac{dX(e^{jw})}{dw}\right|^2 dw = 316\pi$$

7U. Let $X(e^{jw})$ denote the Fourier transform of the signal x[n] depicted in the following figure:



Perform the following calculations without explicitly evaluating $X(e^{jw})$:

a) **Evaluate $X(e^{j0})$**
b) **Find the phase of $X(e^{jw})$**
c) **Evaluate** $\int_{-\pi}^{\pi} X(e^{jw})dw$
d) **Find $X(e^{j\pi})$**
e) **Determine and sketch the signal whose Fourier transform is Pure Imaginary {$X(e^{jw})$}**
f) **Evaluate** $\int_{-\pi}^{\pi} | X(e^{jw}) |^2 \, dw$
g) **Evaluate** $\int_{-\pi}^{\pi} | \frac{dX(e^{jw})}{dw} |^2 \, dw$

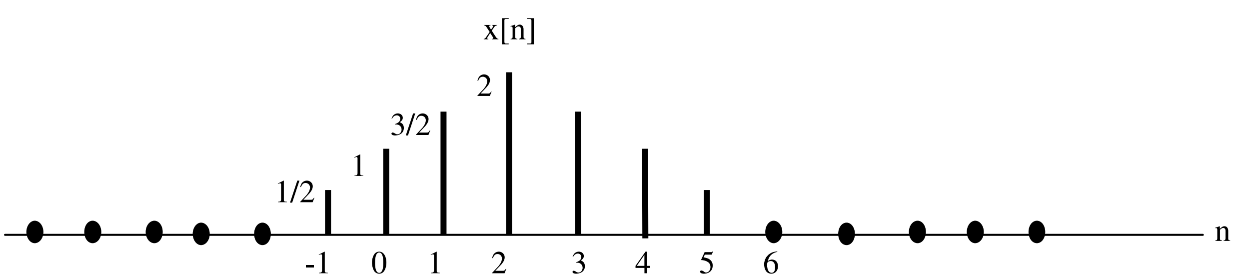
Solution:

8S. Six conditions are listed below:

1) **Real{$X(e^{jw})$} = 0**
2) **Imaginary{$X(e^{jw})$} = 0**
3) **There exist a real p such that $e^{jpw} X(e^{jw})$ is real.**
4) $\int_{-\pi}^{\pi} X(e^{jw})dw$ **is periodic**
5) **Periodic $X(e^{jw})$**
6) **$X(e^{j0}) = 0$**

Which, if any, of the following signals have Fourier transforms that satisfy each of the above conditions:

a) **x[n] Shown in the following diagram**



b) **$x[n] = (1/2)^{|n|}$**
c) **$x[n] = \delta[n - 1] + \delta[n + 3]$**

Solution:

Facts:

1) For Re{$X(e^{jw})$} to be zero, the signal must be real and odd.
2) For Im{$X(e^{jw})$} to be zero, the signal must be real and even.
3) Assume $Y(e^{jw}) = e^{jaw}X(e^{jw})$. Using the time shifting property of the Fourier transform we have y[n]=x[n+a]. If $Y(e^{jw})$ is real, then y[n] is real and even (assuming that x[n] is real). Therefore x[n] has to be symmetric about a.
4) Since $\int_{-\pi}^{\pi} X(e^{jw})dw = 2\pi x[0]$. The given condition is satisfied only if x[0]=0.
5) $X(e^{jw})$ is always periodic with period 2π. Therefore, all signals satisfy this condition.
6) Since $X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n]$, the given condition is satisfied only if samples of the signal add up to zero.

	Cond. #1	Cond. #2	Cond. #3	Cond. #4	Cond. #5	Cond. #6
Part a			Yes		Yes	
Part b		Yes	Yes		Yes	
Part c					Yes	

8U. Six conditions are listed below:

1) **Real{$X(e^{jw})$} = 0**
2) **Imaginary{$X(e^{jw})$} = 0**
3) **There exist a real a such that e^{jaw} $X(e^{jw})$ is real.**
4) $\int_{-\pi}^{\pi} X(e^{jw})dw$ **is periodic**
5) **Periodic $X(e^{jw})$**
6) **$X(e^{j0}) = 0$**

Which, if any, of the following signals have Fourier transforms that satisfy each of the above conditions:

a) **$x[n] = (1/3)^{|n|}$**
b) **$x[n] = \delta[n + 1] + \delta[n - 3]$**

Solution:

9S. Using $X(e^{jw})$ as the Fourier transform of x[n], express the Fourier transform of the following signal in terms of $X(e^{jw})$.

$$y[n] = (n-1)^2 x[n]$$

Solution:

$$Differentiation \Rightarrow nx[n] \xrightarrow{FT} j\frac{dX(e^{jw})}{dw}$$

$$Differentiation \Rightarrow n^2x[n] \xrightarrow{FT} -\frac{d^2X(e^{jw})}{dw^2}$$

$$Therefore \Rightarrow y[n] = [n^2 - 2n + 1)x[n] \Rightarrow Y(e^{jw}) = -\frac{d^2X(e^{jw})}{dw^2} - 2j\frac{dX(e^{jw})}{dw} + X(e^{jw})$$

9U. Given that x[n] has Fourier transform $X(e^{jw})$, express the Fourier transform of the following signal in terms of $X(e^{jw})$:

y[n] = (n^2+ 3n + 5)x[n]

Solution:

10S. Use the inverse Fourier transform integral to determine the inverse Fourier transform of:

$$X(e^{jw}) = \sum_{k=-\infty}^{\infty}\{2\pi\delta(w-2\pi k) + \pi\delta(w-\frac{\pi}{2}-2\pi k) + \pi\delta(w+\frac{\pi}{2}-2\pi k)\}$$

Solution:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \qquad \textit{Synthesis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty}\{2\pi\delta(w-2\pi k) + \pi\delta(w-\frac{\pi}{2}-2\pi k) + \pi\delta(w+\frac{\pi}{2}-2\pi k)\}e^{jwn}dw$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{2\pi\delta(w) + \pi\delta(w-\frac{\pi}{2}) + \pi\delta(w+\frac{\pi}{2})\}e^{jwn}dw$$

$$x[n] = e^{j0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n} = 1 + \cos(\pi n/2)$$

10U. Express the Fourier transform of the following signal in terms of X(e^{jw}):

y [n] = $\frac{x^*[-n] + x[n]}{2}$

Solution: