Signals & Systems - Chapter 3

1S. Continuous-time periodic signal x(t) is real valued and has a fundamental period T=8. The nonzero Fourier series coefficients for x(t) are

$a_1 = a_{-1} = 2$, $a_3 = a_{-3}^* = 4j$.

Express x(t) in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = 2e^{-jw_0 t} + 2e^{jw_0 t} - 4je^{-j3w_0 t} + 4je^{j3w_0 t}$$

Apply Eulers

$$x(t) = 2(2\cos w_0 t) + 4j(2j\sin 3w_0 t) = 4\cos w_0 t - 8\sin 3w_0 t$$

$$w_0 = 2\pi / T = 2\pi / 8 = \pi / 4$$

$$x(t) = 4\cos \pi t/4 - 8\cos(3\pi t/4 - \pi/2)$$

1U. Continuous-time periodic signal x(t) is real valued and has a fundamental period T=12. The nonzero Fourier series coefficients for x(t) are

$a_0 = 4$, $a_2 = a_{-2}^* = j$, $a_3 = a_{-3}^* = -4j$.

Express x(t) in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

Solution:

2S. A discrete-time periodic signal x[n] is real valued and has a fundamental period N=5. The nonzero Fourier series coefficient for x[n] are

$a_0 = 1$, $a_2 = a_{-2}^* = e^{j\pi/4}$, $a_4 = a_{-4}^* = 2e^{j\pi/3}$

Express x[n] in the form

$$x[n] = A_o + \sum_{k=1}^{\infty} A_k \sin(w_k n + \phi_k).$$

Solution:

$$Note: a_{(k+rN)} \text{ where is any integer} \rightarrow a_1 = a_{(1-5)} = a_{-4} = 2e^{-j\pi/3} \quad \& \; a_{-1} = a_{(-1+5)} = a_4 = 2e^{j\pi/3}$$

$$x[n] = \sum_{k=-2}^{2} a_k e^{jkw_0 t} = a_0 + a_{-1}e^{-jw_0 n} + a_1 e^{jw_0 n} + a_{-2}e^{-j2w_0 n} + a_2 e^{j2w_0 n}$$

$$x[n] = 1 + 2e^{-j\pi/3}e^{-jw_0 n} + 2e^{+j\pi/3}e^{jw_0 n} + e^{-j\pi/4}e^{-j2w_0 n} + e^{j\pi/4}e^{j2w_0 n}$$

$$x[n] = 1 + 2\{e^{-j(\pi/3 + w_0 n)} + e^{j(\pi/3 + w_0 n)}\} + \{e^{-j(\pi/4 + j2w_0 n)} + e^{j(\pi/4 + j2w_0 n)}\}$$

Apply Eulers

$$x[n] = 1 + 4\cos(w_0 n + \pi/3) + 2\cos(w_0 n + \pi/4)$$

$$w_0 = 2\pi / N = 2\pi / 5$$

$$x[n] = 1 + 4\sin(w_0 n + 5\pi/6) + 2\sin(w_0 n + 3\pi/4)$$

2U. A discrete-time periodic signal x[n] is real valued and has a fundamental period N=9. The nonzero Fourier series coefficient for x[n] are

$$a_0 = 2,\ a_3 = a_{-3}^* = 3e^{j\pi/2},\ a_4 = a_{-4}^* = 2e^{-j\pi/4}$$

Express x[n] in the form

$$x[n] = A_o + \sum_{k=1}^{\infty} A_k \sin(w_k n + \phi_k).$$

Solution:

3S. For the continuous-time period signal

$$x(t) = 2 + \cos(\frac{2\pi}{3}t) + 4\sin(\frac{5\pi}{3}t)$$

determine the fundamental frequency w_o and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_o t}$$

Solution:

Apply Eulers to the x(t)

$$x(t) = 2 + \frac{1}{2}\left[e^{+j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}\right] - 2j\left[e^{+j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t}\right]$$

$$x(t) = 2 + \frac{1}{2}e^{+j2(\frac{2\pi}{6})t} + \frac{1}{2}e^{-j2(\frac{2\pi}{6})t} - 2je^{+j5(\frac{2\pi}{6})t} + 2je^{-j5(\frac{2\pi}{6})t}$$

$$w_0 = \frac{2\pi}{6}\ \textit{Fundamental Frequency}$$

$a_0=2$; $a_2= a_{-2}=1/2$; $a_5= a_{-5}^*=-2j$;

3U. For the continuous-time period signal

$$x(t) = 3 + 4\cos(\frac{4\pi}{7}t) + 2\sin(\frac{3\pi}{5}t)$$

determine the fundamental frequency w_o and the Fourier series coefficients a_k.

Solution:

4S. Use the Fourier series analysis equation to calculate the coefficients a_k for the continuous-time periodic signal

$$x(t) = \begin{matrix} 1.5 & for & 0 \le t < 1 \\ -1.5 & for & 1 \le t < 2 \end{matrix}$$

with fundamental frequency $w_o = \pi$.

Solution: Continuous-time system, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_o t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad \textit{Fourier Series Synthesis Equation}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_o t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt \quad \textit{Fourier Series Analysis Equation}$$

$$w_0 = \pi \rightarrow \pi T = 2\pi \rightarrow T = 2$$

$$a_k = \frac{1}{2}\int_0^2 x(t)e^{-jkw_o t}dt = \frac{1}{2}\int_0^1 1.5e^{-jkw_o t}dt - \frac{1}{2}\int_1^2 1.5e^{-jkw_o t}dt$$

$$for\ k = 0 \rightarrow a_0 = 0$$

$$for\ k \neq 0$$

$$a_k = \frac{1.5}{2jkw_0}(e^{-jkw_o} - 1 - e^{-j2kw_o} + e^{-jkw_o}) = \frac{1.5}{-2jk\pi}(2e^{-jk\pi} - 1 - e^{-j2k\pi}) = \frac{1.5}{-2jk\pi}(2e^{-jk\pi} - 2)$$

$$for\ even\ k \rightarrow a_k = 0$$

$$for\ odd\ k \rightarrow a_k = \frac{1.5}{-2jk\pi}(2\cos k\pi - j2\sin k\pi - 2) = \frac{3}{jk\pi}$$

4U. Use the Fourier series analysis equation to calculate the coefficients a_k for the continuous-time periodic signal

$$\textbf{x(t) =} \begin{matrix} +1 & for & 0 \le t < 0.5\ m\sec \\ -1 & for & 0.5\ m\sec \le t < 1\ m\sec \end{matrix}$$

with fundamental frequency $w_o = 2000\pi$.

Solution:

5S. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} (\frac{1}{2})^k e^{jk(\frac{2\pi}{50})t}$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk(\frac{2\pi}{50})t}$$

$$x_3(t) = \sum_{k=-100}^{100} j\sin(\frac{k\pi}{2}) e^{jk(\frac{2\pi}{50})t}$$

Use Fourier series properties to help answer the following questions:

a) Which of the three signals is/are real valued?

b) Which of the three signals is/are even?

Solution:

a) if $a_k = a_{-k}^*$ Then the signal x(t) is real otherwise it is not.

Note: conjugate means that (A+jB)* = (A-jB)

$x_1(t)$ Fourier series coefficients are $a_k = (1/2)^K$ for $0 \le k \le 100$ otherwise $a_k=0$

We know that $\{a_k=(1/2)^K\} \neq \{a_{-k}^*=(1/2)^{-K}\}$ therefore $x_1(t)$ is not Real

$x_2(t)$ Fourier series coefficients are $a_k = \cos(k\pi)$ for $-100 \le k \le 100$ otherwise $a_k=0$

We know that $\{a_k= \cos(k\pi)\} = \{a_{-k}^*= \cos(-k\pi)\}$ therefore $x_2(t)$ is Real

$x_3(t)$ Fourier series coefficients are $a_k = j\sin(k\pi/2)$ for $-100 \le k \le 100$ otherwise $a_k=0$

We know that $\{a_k= j\sin(k\pi/2)\} = \{a_{-k}^*= -j\sin(-k\pi/2)\}$ therefore $x_2(t)$ is Real

b) For a signal x(t) to be even its Fourier Series Coefficient a_k must be even
In other words the relationship "x(t)=x(-t) ← → $a_k = a_{-k}$" is true
Which means only $x_2(t)$ is even since only for this function $a_k = a_{-k}$

5U. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sin(2000k\pi t) \quad \textit{where } k \textit{ is an integer}$$
$$x_2(t) = 10\cos(15.294k\pi t) + 10 \quad \textit{where } k \textit{ is real}$$
$$x_3(t) = \sin(2000k\pi t) + j15 \quad \textit{where } k \textit{ is an integer}$$

Use Fourier series properties to help answer the following questions:
a) Which of the three signals is/are real valued?
b) Which of the three signals is/are even?

Solution:

6S. Use the analysis equation to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

$$x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n-4m] + 8\delta[n-1-4m]\}.$$

Solution: For Discrete-time system, we have

$$x[n] = \sum_{k=<N>} a_k e^{jkw_o n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \quad \textit{Fourier Series Synthesis Equation}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_o n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n} \quad \textit{Fourier Series Analysis Equation}$$

a)

first understand the signal $x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n-4m] + 8\delta[n-1-4m]\}$

using the definition of impulse function we can write:

x[n] = 4 for n=4m
8 for n=4m+1
0 otherwise

We see that the signal is periodic with a fundamental period of N=4.
[If you don't see it, just find value for x[0] = 4, x[1] = 8, x[2] = 0, x[3] = 0, x[4] = 4, x[5] = 8, … which repeats every four terms]

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n} = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/4)n} = \frac{1}{4}\left[4e^0 + 8e^{-jk(2\pi/4)}\right] = 1 + 2e^{-jk(\pi/2)}$$

Therefore:

$a_0 = 3$, $a_1 = 1 - 2j$, $a_2 = -1$, $a_3 = 1 + 2j$

6U. Use the analysis equation to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

$$x[n] = \sum_{m=-\infty}^{\infty} \{3\delta[n-5m] + 2\delta[n-2-5m]\}.$$

Solution:

7S. Let x[n] be a real and odd periodic signal with period N=7 and Fourier coefficient a_k. Given that

$a_{15} = j$, $a_{16} = 2j$, $a_{17} = 3j$.

Determine the values of a_0, a_{-1}, a_{-2} and a_{-3}.

Solution:

Using the properties of Fourier Series we could state:

1) Period with period N=7 → $a_k = a_{k+7n}$

$a_1 = a_{1+2*7} = a_{15} = j$
$a_2 = a_{2+2*7} = a_{16} = 2j$
$a_3 = a_{3+2*7} = a_{17} = 3j$

2) real and odd x(t) → a_k is purely imaginary and odd ($a_k = -a_{-k}$)

$a_0 = 0$
$a_{-2} = -a_2 = -2j$
$a_{-3} = -a_3 = -3j$

7U. Let x[n] be a real and odd periodic signal with period N=9 and Fourier coefficient a_k. Given that

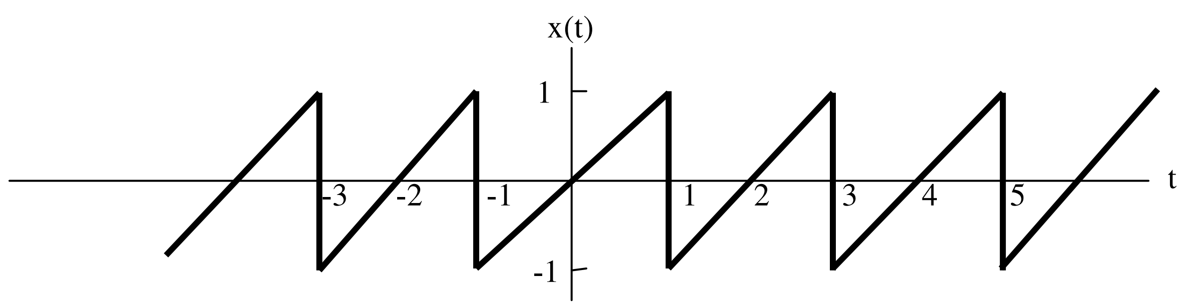
$a_{15} = 2j$, $a_{16} = 3j$ $a_{17} = -4j$.

Determine the values of a_0, a_{-1}, a_{-2} and a_{-3}.

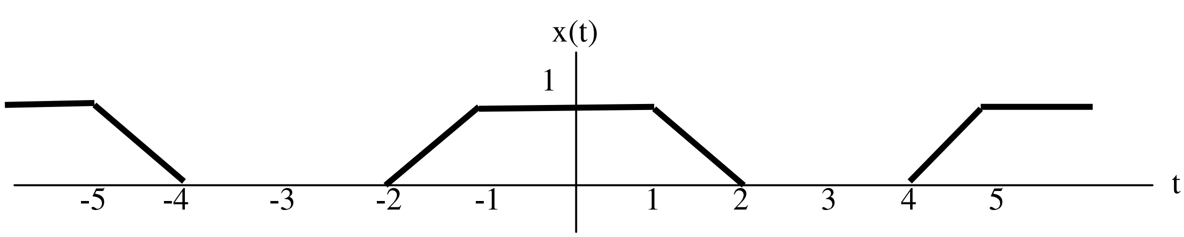
Solution:

8S. Determine the Fourier series representations for the following signals:

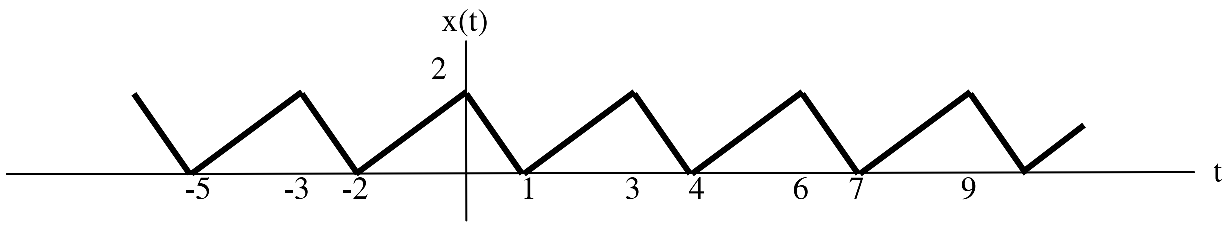
a) Each x(t) illustrated in the following figure



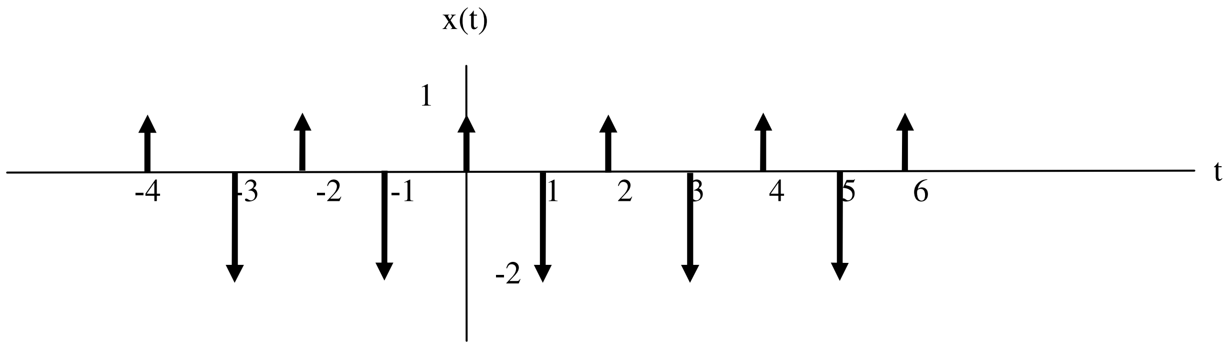
b) Each x(t) illustrated in the following figure



c) Each x(t) illustrated in the following figure



d) Each x(t) illustrated in the following figure



e) x(t) is a periodic signal with period 2 and
$x(t) = e^{-t}$ for $-1 < t < 1$

Solution:

a)

$$T = 2 \rightarrow w_0 = 2\pi / T = \pi$$

$$x(t) = t$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt = \frac{1}{2}\int_{-1}^{1} te^{-jkw_0t}dt$$

$$a_0 = 0$$

$$Integration\ by\ part \rightarrow \int udv = uv - \int vdu$$

$$a_k = \frac{j(-1)^k}{k\pi} \quad for\ k \neq 0$$

b)

$$T = 6 \rightarrow w_0 = 2\pi / T = \pi / 3$$
$$x(t) = t + 2 \quad for -2 < t < -1$$
$$1 \quad for -1 < t < +1$$
$$2 - t \quad for +1 < t < +2$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}\,dt$$
$$a_k = \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-jkw_0t}\,dt + \int_{-1}^{1}e^{-jkw_0t}\,dt + \int_{1}^{2}(2-t)e^{-jkw_0t}\,dt\right]$$
$$a_0 = 1/2$$
$$a_k = 0 \qquad for\ even\ k$$
$$= \frac{6}{\pi^2 k^2}\sin(\frac{\pi k}{2})\sin(\frac{\pi k}{6}) \quad for\ odd\ k$$

c)

$$T = 3 \rightarrow w_0 = 2\pi / T = 2\pi / 3$$
$$x(t) = t + 2 \quad for -2 < t < 0$$
$$2 - 2t \quad for\ 0 < t < +1$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}\,dt$$
$$a_k = \frac{1}{3}\left[\int_{-2}^{0}(t+2)e^{-jkw_0t}\,dt + \int_{0}^{1}(2-2t)e^{-jkw_0t}\,dt\right]$$
$$a_0 = 1$$
$$a_k = \frac{3j}{2\pi^2 k^2}\left[e^{jk2\pi/3}\sin(k2\pi/3) + 2e^{jk\pi/3}\sin(k\pi/3)\right] \quad for\ k \neq 0$$

d)

$$T = 2 \rightarrow w_0 = 2\pi / T = \pi$$
$$x(t) = 1 \quad for\ t = 0$$
$$-2 \quad for\ t = 1$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}\,dt$$
$$a_0 = -1/2$$
$$a_k = \frac{1}{2} - (-1)^k \quad for\ k \neq 0$$

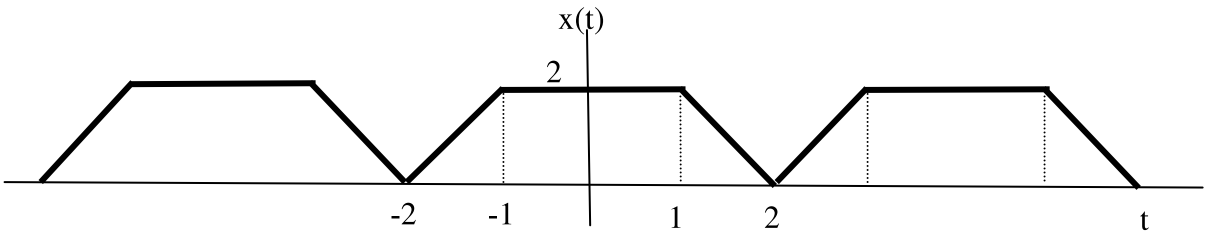
e)

$$T = 2 \rightarrow w_0 = 2\pi / T = \pi$$

$$x(t) = e^{-t} \quad for \quad -1 < t < 1$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt = \int_{-1}^{1} e^{-t}e^{-jk\pi t}dt = \frac{-1}{1+jk\pi}\left(e^{-(1+jk\pi)t}\right|_{-1}^{1} = \frac{e^{(1+jk\pi)} - e^{-(1+jk\pi)}}{1+jk\pi} \quad for\ all\ k$$

8U. Determine the Fourier Series representation for the signal shown in the following figure:



Solution:

9S. A discrete-time periodic signal x[n] is real valued and has fundamental period N=5. The nonzero Fourier series coefficients for x[n] are

$a_0 = 2$, $a_2 = a_{-2}^* = 2e^{j\pi/6}$, $a_4 = a_{-4}^* = e^{j\pi/3}$

Express x[n] in the form

x[n] = $A_o + \sum_{k=1}^{\infty} A_k \sin(w_k n + \phi_k)$.

Solution:

$$w_0 = 2\pi / N = 2\pi / 5$$

$$x[n] = \sum_{k=0}^{4} a_k e^{jkw_0n} = a_0 + a_{-2}e^{-j2(2\pi/5)n} + a_2 e^{j2(2\pi/5)n} + a_{-4}e^{-j4(2\pi/5)n} + a_4 e^{j4(2\pi/5)n}$$

$$x[n] = 2 + 2e^{-j\pi/6}e^{-j(4\pi/5)n} + 2e^{j\pi/6}e^{j(4\pi/5)n} + e^{-j\pi/3}e^{-j(8\pi/5)n} + e^{j\pi/3}e^{j(8\pi/5)n} \quad =$$

$$x[n] = 2 + 4\cos[(4\pi/5)n + \pi/6] + 4\cos[(8\pi/5)n + \pi/3]$$

$$add\ \pi/2$$

$$x[n] = 2 + 4\cos[(4\pi/5)n + 2\pi/3] + 4\cos[(8\pi/5)n + 5\pi/6]$$

9U. A discrete-time periodic signal x[n] is real valued and has fundamental period N=7. The nonzero Fourier series coefficients for x[n] are

$a_0 = 2$, $a_1 = a_{-1}^* = e^{j\pi/3}$, $a_3 = a_{-3}^* = 3e^{j\pi}$

Express x[n] in the form

x[n] = $A_o + \sum_{k=1}^{\infty} A_k \sin(w_k n + \phi_k)$.

Solution:

10S. Determine the Fourier series coefficients for the following discrete-time periodic signal. Plot the magnitude and phase of each set of coefficients a_k.

x[n] = sin(2πn/3)cos(πn/2)

Solution:

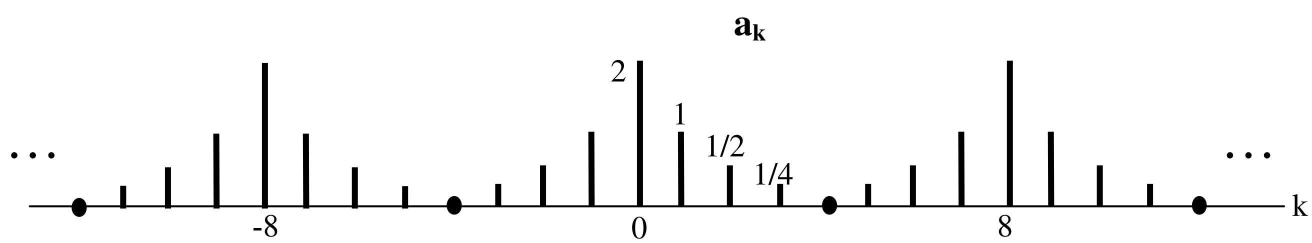
$$first\ term\ w_0 = 2\pi/N = 2\pi/3 \Rightarrow N_1 = 3$$
$$second\ term\ w_0 = 2\pi/N = \pi/2 \Rightarrow N_2 = 4$$
$$Therefore\ periond\ for\ X[n] \Rightarrow N = 12$$
$$a_k = \frac{1}{N}\sum_{n=0}^{11} x[n]e^{-jkw_0 n} = \frac{1}{12}\sum_{n=0}^{11}\sin(2\pi n/3)\cos(\pi n/2)e^{-jkw_0 n}$$

10U. Determine the Fourier series coefficients for the following discrete-time periodic signal. Plot the magnitude and phase of each set of coefficients a_k.

$x[n] = \sin(6\pi n/7)e^{(j\pi n/10)}$

Solution:

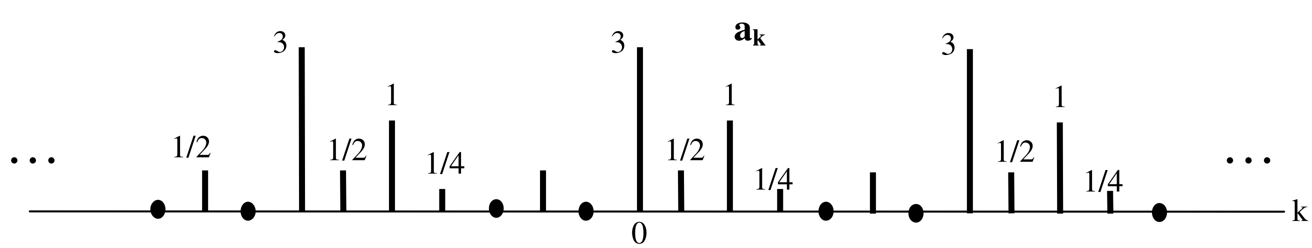
11S. The signal represented by the following Fourier series coefficients is a periodic with period 8. Determine the signal x[n].



Solution:

$$w_0 = 2\pi/8 = \pi/4$$
$$x[n] = \sum_{k=-4}^{3} a_k e^{jkw_0 n} = 2 + e^{j(\pi/4)n} + e^{-j(\pi/4)n} + \frac{1}{2}e^{j(\pi/2)n} + \frac{1}{2}e^{-j(\pi/2)n} + \frac{1}{4}e^{j(3\pi/4)n} + \frac{1}{4}e^{-j(3\pi/4)n} + 0$$
$$x[n] = 2 + 2\cos(\pi n/4) + \cos(\pi n/2) + (1/2)\cos(3\pi n/4)$$

11U. The signal represented by the following Fourier series coefficients is a periodic with period 7. Determine the signal x[n].



Solution: