1S. Continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T=8$. The nonzero Fourier series coefficients for $x(t)$ are $a_1 = a_{-1} = 2$, $a_3 = a_{-3}^* = 4j$.

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

Solution:

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{jkw_0 t} = 2e^{-j\pi/2} + 2e^{j\pi/2} - 4je^{-j3\pi/2} + 4je^{j3\pi/2}$$

Apply Eulers

$$x(t) = 2(2\cos w_0 t) + 4j(2 j \sin 3w_0 t) = 4\cos w_0 t - 8\sin 3w_0 t$$

$$w_0 = 2\pi / T = 2\pi / 8 = \pi / 4$$

$$x(t) = 4\cos \pi / 4 - 8\cos(3\pi / 4 - \pi / 2)$$

1U. Continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T=12$. The nonzero Fourier series coefficients for $x(t)$ are $a_0 = 4$, $a_2 = a_{-2}^* = j$, $a_3 = a_{-3}^* = -4j$.

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

Solution:

$$x(t) = \sum_{k=\infty}^{\infty} a_k e^{jkw_0 t} = 0 + a_{-1} e^{-j\pi/4} + a_2 e^{j\pi/4} + a_{-2} e^{-j2\pi/4} + a_4 e^{j2\pi/4}$$

Note: $a_{k+N}$ where is any integer $\rightarrow a_1 = a_{-1} = \cdots = a_{-4} = 2e^{-j\pi/3}$ & $a_{-1} = a_{-1+5} = a_4 = 2e^{j\pi/3}$

$$x[n] = 0 + a_{-1} e^{-j\pi/4} + a_{-2} e^{-j2\pi/4} + a_2 e^{j2\pi/4}$$

$$x[n] = 1 + 2e^{-j\pi/3} e^{-j\pi/4} + 2e^{j\pi/3} e^{j\pi/4} + e^{-j\pi/4} e^{-j2\pi/4} + e^{j\pi/4} e^{j2\pi/4}$$

$$x[n] = 1 + 2\{e^{-j(\pi/3+w_0/3)} + e^{j(\pi/3+w_0/3)}\} + \{e^{-j(\pi/4 + j2\pi)} + e^{j(\pi/4 + j2\pi)}\}$$

Apply Eulers

$$x[n] = 1 + 4\cos(w_0 n + \pi / 3) + 2\cos(w_0 n + \pi / 4)$$

$$w_0 = 2\pi / N = 2\pi / 5$$

$$x[n] = 1 + 4\sin(w_0 n + 5\pi / 6) + 2\sin(w_0 n + 3\pi / 4)$$
2U. A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period $N=9$. The nonzero Fourier series coefficient for $x[n]$ are

$$a_0 = 2, \quad a_3 = a_{-3}^* = 3e^{j\pi/2}, \quad a_4 = a_{-4}^* = 2e^{-j\pi/4}$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(w_n n + \phi_k).$$

Solution:

3S. For the continuous-time period signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3} t\right) + 4 \sin\left(\frac{5\pi}{3} t\right)$$

determine the fundamental frequency $w_o$ and the Fourier series coefficients $a_k$ such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_o t}$$

Solution: Apply Eulers to the $x(t)$

$$x(t) = 2 + \frac{1}{2} \left[ e^{j\frac{2\pi}{3}} + e^{-j\frac{2\pi}{3}} \right] - 2j \left[ e^{j\frac{5\pi}{3}} - e^{-j\frac{5\pi}{3}} \right]$$

$$w_o = \frac{2\pi}{6} \quad \text{Fundamental Frequency}$$

$$a_0 = 2; \quad a_2 = a_{-2} = 1/2; \quad a_5 = a_{-5}^* = -2;$$

3U. For the continuous-time period signal

$$x(t) = 3 + 4 \cos\left(\frac{4\pi}{7} t\right) + 2 \sin\left(\frac{3\pi}{5} t\right)$$

determine the fundamental frequency $w_o$ and the Fourier series coefficients $a_k$.

Solution:

4S. Use the Fourier series analysis equation to calculate the coefficients $a_k$ for the continuous-time periodic signal

$$x(t) = \begin{cases} 1.5 & \text{for} \quad 0 \leq t < 1 \\ -1.5 & \text{for} \quad 1 \leq t < 2 \end{cases}$$

with fundamental frequency $w_o = \pi$.

Solution: Continuous-time system, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_o t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad \text{Fourier Series Synthesis Equation}$$

$$a_k = \frac{1}{T} \int_{T} x(t) e^{-jk(2\pi/T)t} dt = \frac{1}{T} \int_{T} x(t) e^{-jk(2\pi/T)t} dt \quad \text{Fourier Series Analysis Equation}$$
\( w_0 = \pi \rightarrow \pi T = 2\pi \rightarrow T = 2 \)

\[ a_k = \frac{1}{2} \int_0^1 x(t) e^{-jkw_0t} \, dt = \frac{1}{2} \int_0^1 1.5 e^{-jkw_0t} \, dt - \frac{1}{2} \int_1^2 1.5 e^{-jkw_0t} \, dt \]

For \( k = 0 \rightarrow a_0 = 0 \)

For \( k \neq 0 \)

\[ a_k = \frac{1.5}{2jkw_0} (e^{-jkw_0} - 1 - e^{-j2kw_0} + e^{-jkw_0}) = \frac{1.5}{2jk\pi} (2e^{-jk\pi} - 1 - e^{-j2k\pi}) = \frac{1.5}{2jk\pi} (2e^{-jk\pi} - 2) \]

For even \( k \rightarrow a_k = 0 \)

For odd \( k \rightarrow a_k = \frac{1.5}{2jk\pi} (2\cos k\pi - j2\sin k\pi - 2) = \frac{3}{jk\pi} \)

4U. Use the Fourier series analysis equation to calculate the coefficients \( a_k \) for the continuous-time periodic signal

\[ x(t) = \begin{cases} 1 & \text{for } 0 \leq t < 0.5 \text{ m sec} \\ -1 & \text{for } 0.5 \text{ m sec} \leq t < 1 \text{ m sec} \end{cases} \]

with fundamental frequency \( w_0 = 2000\pi \).

Solution:

5S. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

\[ x_1(t) = \sum_{k=0}^{100} \left( \frac{1}{2} \right)^k e^{j\frac{2\pi}{50}kt} \]

\[ x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{j\frac{2\pi}{50}kt} \]

\[ x_3(t) = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) e^{j\frac{2\pi}{50}kt} \]

Use Fourier series properties to help answer the following questions:

a) Which of the three signals is/are real valued?

b) Which of the three signals is/are even?

Solution:

a) if \( a_k = a_{-k}^* \) Then the signal \( x(t) \) is real otherwise it is not.

Note: conjugate means that \((A+jB)^* = (A-jB)\)

\( x_1(t) \) Fourier series coefficients are \( a_k = (1/2)^k \) for \( 0 \leq k \leq 100 \) otherwise \( a_k = 0 \)

We know that \( \{a_k=(1/2)^k\} \neq \{a_k^*=(1/2)^{-k}\} \) therefore \( x_1(t) \) is not Real

\( x_2(t) \) Fourier series coefficients are \( a_k = \cos(k\pi) \) for \( -100 \leq k \leq 100 \) otherwise \( a_k = 0 \)

We know that \( \{a_k=\cos(k\pi)\} = \{a_k^*=\cos(-k\pi)\} \) therefore \( x_2(t) \) is Real

\( x_3(t) \) Fourier series coefficients are \( a_k = j\sin(k\pi/2) \) for \( -100 \leq k \leq 100 \) otherwise \( a_k = 0 \)

We know that \( \{a_k=j\sin(k\pi/2)\} = \{a_k^*=-j\sin(-k\pi/2)\} \) therefore \( x_3(t) \) is Real
b) For a signal \( x(t) \) to be even its Fourier Series Coefficient \( a_k \) must be even
In other words the relationship \( x(t) = x(-t) \) \( \leftrightarrow \) \( a_k = a_{-k} \) is true
Which means only \( x_2(t) \) is even since only for this function \( a_k = a_{-k} \)

5U. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

\[
\begin{align*}
x_1(t) &= \sin(2000k\pi) \quad \text{where } k \text{ is an integer} \\
x_2(t) &= 10 \cos(15.294k\pi) + 10 \quad \text{where } k \text{ is real} \\
x_3(t) &= \sin(2000k\pi) + j15 \quad \text{where } k \text{ is an integer}
\end{align*}
\]

Use Fourier series properties to help answer the following questions:

a) Which of the three signals is/are real valued?
b) Which of the three signals is/are even?

Solution:

6U. Use the analysis equation to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

\[
x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n - 4m] + 8\delta[n - 1 - 4m]\}.
\]

Solution: For Discrete-time system, we have

\[
x[n] = \sum_{k\left< N \right>} a_k e^{jkw_n} = \sum_{k\left< N \right>} a_k e^{j(2\pi/N)n} \quad \text{Fourier Series Synthesis Equation}
\]

\[
a_k = \frac{1}{N} \sum_{n\left< N \right>} x[n] e^{-jkw_n} = \frac{1}{N} \sum_{n\left< N \right>} x[n] e^{-j(2\pi/N)n} \quad \text{Fourier Series Analysis Equation}
\]

a)

first understand the signal \( x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n - 4m] + 8\delta[n - 1 - 4m]\} \)

using the definition of impulse function we can write:

\[
x[n] = 4 \quad \text{for } n=4m
\]

\[
8 \quad \text{for } n=4m+1
\]

\[
0 \quad \text{otherwise}
\]

We see that the signal is periodic with a fundamental period of \( N=4 \).

[If you don’t see it, just find value for \( x[0] = 4, x[1] = 8, x[2] = 0, x[3] = 0, x[4] = 4, x[5] = 8, ... \) which repeats every four terms]

\[
a_k = \frac{1}{N} \sum_{n\left< N \right>} x[n] e^{-j(2\pi/N)n} = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-j(2\pi/4)n} = \frac{1}{4} \left[ 4e^0 + 8e^{-j(2\pi/4)} \right] = 1 + 2e^{-j(\pi/2)}
\]

Therefore:

\[
a_0 = 3, \quad a_1 = 1 - 2j, \quad a_2 = -1, \quad a_3 = 1 + 2j
\]

6U. Use the analysis equation to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

\[
x[n] = \sum_{m=-\infty}^{\infty} \{3\delta[n - 5m] + 2\delta[n - 2 - 5m]\}.
\]

Solution:
7S. Let $x[n]$ be a real and odd periodic signal with period $N=7$ and Fourier coefficient $a_k$. Given that $a_{15} = j$, $a_{16} = 2j$, $a_{17} = 3j$.

Determine the values of $a_0$, $a_{-1}$, $a_2$ and $a_3$.

Solution:

Using the properties of Fourier Series we could state:
1) Period with period $N=7 \rightarrow a_k = a_{k+7n}$
   
   $a_1 = a_{1+2\cdot7} = a_{15} = j$
   $a_2 = a_{2+2\cdot7} = a_{16} = 2j$
   $a_3 = a_{3+2\cdot7} = a_{17} = 3j$

2) real and odd $x(t) \rightarrow a_k$ is purely imaginary and odd ($a_k = -a_{-k}$)
   
   $a_0 = 0$
   $a_{-2} = -a_2 = -2j$
   $a_{-3} = -a_3 = -3j$

7U. Let $x[n]$ be a real and odd periodic signal with period $N=9$ and Fourier coefficient $a_k$. Given that $a_{15} = 2j$, $a_{16} = 3j$, $a_{17} = -4j$.

Determine the values of $a_0$, $a_{-1}$, $a_2$ and $a_3$.

Solution:

8S. Determine the Fourier series representations for the following signals:
   a) Each $x(t)$ illustrated in the following figure

   ![Figure 1](image1.jpg)

   $x(t)$

   $t$

   -3 -2 -1 1 2 3 4 5

   b) Each $x(t)$ illustrated in the following figure

   ![Figure 2](image2.jpg)

   $x(t)$

   $t$

   -5 -4 -3 -2 -1 1 2 3 4 5

   c) Each $x(t)$ illustrated in the following figure

   ![Figure 3](image3.jpg)

   $x(t)$

   $t$

   -5 -4 -3 -2 -1 1 2 3 4 5
d) Each $x(t)$ illustrated in the following figure

\[ x(t) = e^{-t} \quad \text{for} \quad -1 < t < 1 \]

Solution:

a) 
\[ T = 2 \rightarrow w_0 = \frac{2\pi}{T} = \pi \]
\[ x(t) = t \]
\[ a_k = \frac{1}{T} \int_{-1}^{1} x(t) e^{-jkw_0 t} \, dt = \frac{1}{2} \int_{-1}^{1} te^{-jkw_0 t} \, dt \]
\[ a_0 = 0 \]

Integration by part \( \rightarrow \int u dv = uv - \int v du \)
\[ a_k = \frac{j(-1)^k}{k\pi} \quad \text{for} \ k \neq 0 \]
b) 
\[ T = 6 \rightarrow w_0 = 2\pi / T = \pi / 3 \]
\[ x(t) = t + 2 \quad \text{for} \ -2 < t < -1 \]
\[ 1 \quad \text{for} \ -1 < t < +1 \]
\[ 2 - t \quad \text{for} \ +1 < t < +2 \]
\[ a_k = \frac{1}{T} \int x(t) e^{-jkw_0t} \, dt \]
\[ a_k = \frac{1}{6} \left[ \int_{-1}^{1} (t + 2) e^{-jkw_0t} \, dt + \int_{-1}^{1} e^{-jkw_0t} \, dt + \int_{1}^{2} (2 - t) e^{-jkw_0t} \, dt \right] \]
\[ a_0 = 1/2 \]
\[ a_k = 0 \quad \text{for even} \ k \]
\[ = \frac{6}{\pi^2 k^2} \sin\left(\frac{\pi k}{2}\right) \sin\left(\frac{\pi k}{6}\right) \quad \text{for odd} \ k \]

c) 
\[ T = 3 \rightarrow w_0 = 2\pi / T = 2\pi / 3 \]
\[ x(t) = t + 2 \quad \text{for} \ -2 < t < 0 \]
\[ 2 - 2t \quad \text{for} \ 0 < t < +1 \]
\[ a_k = \frac{1}{T} \int x(t) e^{-jkw_0t} \, dt \]
\[ a_k = \frac{1}{3} \left[ \int_{-2}^{0} (t + 2) e^{-jkw_0t} \, dt + \int_{0}^{1} (2 - 2t) e^{-jkw_0t} \, dt \right] \]
\[ a_0 = 1 \]
\[ a_k = \frac{3j}{2\pi^2 k^2} \left[ e^{jk\pi/3} \sin(k\pi/3) + 2e^{jk\pi/3} \sin(k\pi / 3) \right] \quad \text{for} \ k \neq 0 \]

d) 
\[ T = 2 \rightarrow w_0 = 2\pi / T = \pi \]
\[ x(t) = 1 \quad \text{for} \ t = 0 \]
\[ -2 \quad \text{for} \ t = 1 \]
\[ a_k = \frac{1}{T} \int x(t) e^{-jkw_0t} \, dt \]
\[ a_0 = -1/2 \]
\[ a_k = \frac{1}{2} - (-1)^k \quad \text{for} \ k \neq 0 \]
8U. Determine the Fourier Series representation for the signal shown in the following figure:

![Signal Diagram]

Solution:

9S. A discrete-time periodic signal $x[n]$ is real valued and has fundamental period $N=5$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 2, \ a_2 = a_2^* = 2e^{j\pi/6}, \ a_4 = a_4^* = e^{j\pi/3}$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(w_n n + \phi_k).$$

Solution:

$$w_0 = 2\pi / N = 2\pi / 5$$

$$x[n] = \sum_{k=0}^{4} a_k e^{jkw_0 n} = a_0 + a_2 e^{-j(2\pi/5)n} + a_2 e^{j(2\pi/5)n} + a_4 e^{-j(4\pi/5)n} + a_4 e^{j(4\pi/5)n}$$

$$x[n] = 2 + 2e^{-j\pi/6} e^{-j(4\pi/5)n} + 2e^{j\pi/6} e^{j(4\pi/5)n} + e^{-j\pi/3} e^{-j(8\pi/5)n} + e^{j\pi/3} e^{j(8\pi/5)n}$$

$$x[n] = 2 + 4 \cos[(4\pi/5)n + \pi/6] + 4 \cos[(8\pi/5)n + \pi/3]$$

$$x[n] = 2 + 4 \cos[(4\pi/5)n + 2\pi/3] + 4 \cos[(8\pi/5)n + 5\pi/6]$$

9U. A discrete-time periodic signal $x[n]$ is real valued and has fundamental period $N=7$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 2, \ a_1 = a_1^* = e^{j\pi/3}, \ a_3 = a_3^* = 3e^{j\pi}$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(w_n n + \phi_k).$$

Solution:
\[ x[n] = \sin(2\pi n/3)\cos(\pi n/2) \]

**Solution:**

First term \( w_0 = 2\pi / N = 2\pi / 3 \Rightarrow N_1 = 3 \)

Second term \( w_0 = 2\pi / N = \pi / 2 \Rightarrow N_2 = 4 \)

Therefore period for \( X[n] \Rightarrow N = 12 \)

\[ a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-j\omega_0 n} = \frac{1}{12} \sum_{n=0}^{11} \sin(2\pi n/3)\cos(\pi n/2)e^{-j\omega_0 n} \]

10U. Determine the Fourier series coefficients for the following discrete-time periodic signal. Plot the magnitude and phase of each set of coefficients \( a_k \).

\[ x[n] = \sin(6\pi n/7)e^{jn\pi/10} \]

**Solution:**

11S. The signal represented by the following Fourier series coefficients is a periodic with period 8. Determine the signal \( x[n] \).

**Solution:**

\( w_0 = 2\pi / 8 = \pi / 4 \)

\[ x[n] = \sum_{k=-4}^{3} a_k e^{j\omega_0 n} = 2 + e^{j(\pi/4)n} + e^{-j(\pi/4)n} + \frac{1}{2} e^{j(\pi/2)n} - \frac{1}{2} e^{-j(\pi/2)n} + \frac{1}{4} e^{j(3\pi/4)n} - \frac{1}{4} e^{-j(3\pi/4)n} \]

\[ x[n] = 2 + 2\cos(\pi n/4) + \cos(\pi n/2) + (1/2)\cos(3\pi n/4) \]

11U. The signal represented by the following Fourier series coefficients is a periodic with period 7. Determine the signal \( x[n] \).

**Solution:**