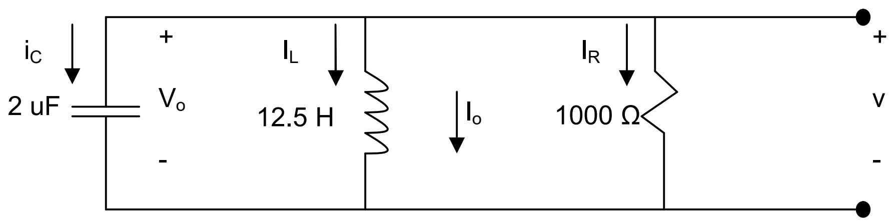
Fundamentals of Electrical Circuits - Chapter 8

1) The resistance, inductance, and capacitance is a parallel RLC circuit are 1000Ω, 12.5 H, and 2 uF, respectively.
a) Calculate the roots of the characteristic equation that describe the voltage response of the circuit.
b) Will the response be over-, under-, or critically damped?
c) What value of R will yield a damped frequency of 120 krad/s?
d) What are the roots of characteristic equation for the value of R found in (c)?
e) What value of R will result in a critically damped response?

Solution:



a) Calculate the roots of the characteristic equation that describe the voltage response of the circuit.

$$\alpha = \frac{1}{2RC} = \frac{1}{2*1000*2*10^{-6}} = 250\ rad/\sec \quad Neper\ Frequency$$

$$w_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{12.5*2*10^{-6}}} = 200 rad/\sec \quad Resonant\ Radian\ Frequency$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2} = -100\ rad/\sec \quad Characteristic\ Root\ or\ Complex\ Frequecy\ 1$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - w_0^2} = -400\ rad/\sec \quad Characteristic\ Root\ or\ Complex\ Frequecy\ 2$$

b) Will the response be over-, under-, or critically damped?
$w_0^2 < \alpha^2 \rightarrow$ Overdamped

c) What value of R will yield a damped frequency of 120 krad/s?

$$w_d = \sqrt{\omega_0^2 - \alpha^2}\ rad/\sec \quad Damped\ Radian\ Frequency$$

$$120 = \sqrt{(200)^2 - (\frac{1}{2R*2*10^{-6}})^2} \Rightarrow R = 1562.5\Omega$$

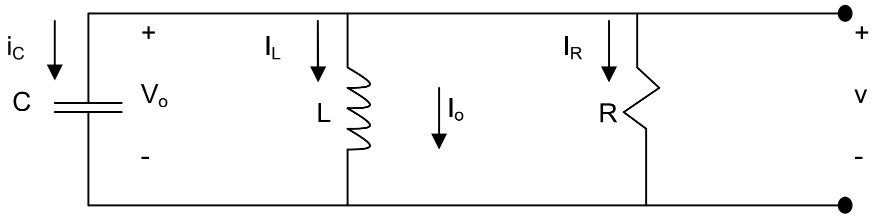
d) What are the roots of characteristic equation for the value of R found in (c)?

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -160 \pm j120 rad/\sec \quad Characteristic\ Roots$$

e) What value of R will result in a critically damped response?
Critically damped response requires $w_0^2 = \alpha^2$

$$(\frac{1}{2RC})^2 = \frac{1}{LC} \Rightarrow (\frac{1}{2*R*2*10^{-6}})^2 = \frac{1}{12.5*2*10^{-6}} \Rightarrow R = 1250\Omega$$

2) The initial voltage on the 0.1 uF capacitor in the following circuit is 24 V. The initial current in the inductor is zero. The voltage response for t ≥ 0 is {$v(t) = -8e^{-250t} + 32e^{-1000t}$ V}



a) Determine the numerical values of R, L, α, w_o.
b) Calculate $i_R(t)$, $i_L(t)$ and $i_c(t)$ for $t \geq 0^+$.

Solution:

a) Determine the numerical values of R, L, α, w_o.
This is RLC Natural Response "$v(t) = -8e^{-250t} + 32e^{-1000t}$ V"

For RLC Natural Response can have any of the following three form

$$Overdamped(\omega_0^2 < \alpha^2) \Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$Critically\ Damped((\omega_0^2 = \alpha^2) \Rightarrow v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$
$$Underdamped(\omega_0^2 > \alpha^2) \Rightarrow v(t) = B_1 e^{-\alpha t}\cos w_d t + B_2 e^{-\alpha t}\sin w_d t$$

From value v(t) given then it is overdamped response →

$$Overdamped(\omega_0^2 < \alpha^2) \Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = -8e^{-250t} + 32e^{-1000t}$$

s_1=-250 ; s_2=-1000

$$\alpha = \frac{1}{2RC} = \frac{5*10^6}{R}\ rad/\sec$$
$$w_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{5*10^6}{L}}\ rad/\sec$$

→ α=-1250 rad/sec; R=8 kΩ; w_0=500 rad/sec; L=40 H

$$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2} = -250\ rad/\sec$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - w_0^2} = -1000\ rad/\sec$$

b) Calculate $i_R(t)$, $i_L(t)$ and $i_c(t)$ for $t \geq 0^+$.

$$i_R = \frac{v(t)}{R} = -1e^{-250t} + 4e^{-1000t} mA \quad t \geq 0^+$$
$$i_C = C\frac{dv(t)}{dt} = 0.2e^{-250t} - 3.2e^{-1000t} mA \quad t \geq 0^+$$
$$i_L = -(i_R + i_C) = 0.8e^{-250t} - 0.8e^{-1000t} mA \quad t \geq 0^+$$

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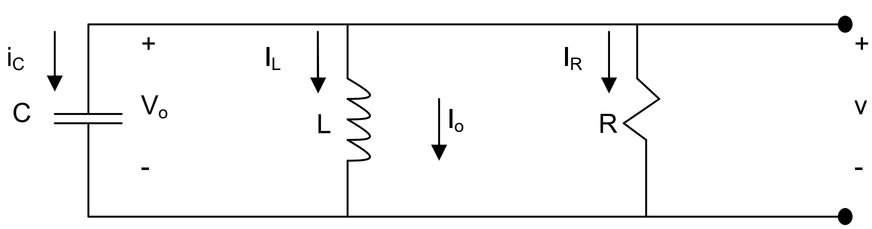
3) The natural voltage response of the following circuit is {$v(t) = 100e^{-20,000t}$[cos(15,000t – 2sin(15,000t) V for $t \geq 0$}.



When the capacitor is 0.04 uF, Find:

a) L **b) R** **c) V_o** **d) I_o** **e) $i_L(t)$**

Solution:

a) L

For RLC Natural Response can have any of the following three form

$$Overdamped(\omega_0^2 < \alpha^2) \Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$Critically\ Damped((\omega_0^2 = \alpha^2) \Rightarrow v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$Underdamped(\omega_0^2 > \alpha^2) \Rightarrow v(t) = B_1 e^{-\alpha t}\cos w_d t + B_2 e^{-\alpha t}\sin w_d t$$

From value v(t) given then it is underdamped response →

$$v(t) = e^{-\alpha t}(B_1 \cos w_d t + B_2 \sin w_d t) = e^{-20,000t}(100\cos 15,000t - 200\sin 15,000t) \quad t \ge 0$$

$$\alpha = 20,000 \quad rad/\sec; \quad \omega_d = 15,000 \quad rad/\sec$$

$$\omega_d = \sqrt{\omega_0{}^2 - \alpha^2} \Rightarrow 15,000 = \sqrt{\omega_0{}^2 - 20,000^2} \Rightarrow \omega_0 = 625x10^6$$

$$\omega_0 = \frac{1}{LC} = \frac{1}{L*0.04*10^{-6}} = 625x10^6 \Rightarrow L = 40mH$$

b) R

$$\alpha = \frac{1}{2RC} \Rightarrow 20,000 = \frac{1}{2R*0.04*10^{-6}} \Rightarrow R = 625\Omega$$

c) V_o

$$Vo = v(0) = 10v$$

d) I_o

$$Io = i_L(0) = -i_R(0) - i_C(0)$$

$$i_R(0) = \frac{Vo}{R} = \frac{100}{625} = 160\ mA$$

$$\frac{dv(t)}{dt} = 100\{e^{-20,000t}[-15,000\sin 15,000t - 30,000\cos 15,000t] - 20,000e^{-20,000t}[\cos 15,000t - 2\sin 15,000t]$$

$$i_C(t) = C\frac{dv}{dt}(0) = (0.4x10^{-6})\{100\{1(-30,000) - 20,000\} = -200\ mA$$

$$Io = i_L(0) = -i_R(0) - i_C(0) = 200 - 160 = 40\ mA$$

e) $i_L(t)$

$$i_L(t) = -i_R(t) - i_C(t)$$

$$i_C(t) = C\frac{dv}{dt}(t) = 0.1e^{-20,000t}[\sin 15,000t - 2\cos 15,000t]\ A$$

$$i_R(t) = \frac{v(t)}{R} = 0.16e^{-20,000t}[\cos 15,000t - 2\sin 15,000t]\ A$$

$$i_L(t) = -i_R(t) - i_C(t) = e^{-20,000t}[40\cos 15,000t - 220\sin 15,000t]\ mA \quad t \ge 0$$

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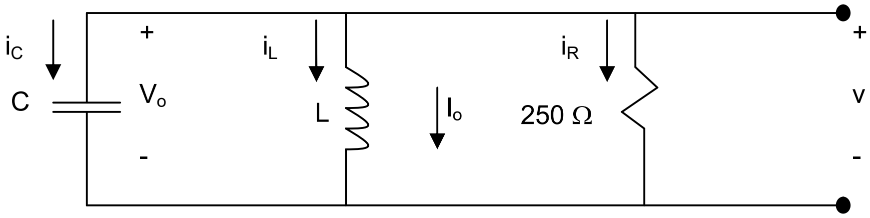
4) The initial value of the voltage v in the following circuit is 15 V. and the initial value of the capacitor current, $i_C(0^+)$, is 45 mA. The expression for the capacitor current is known to be

$$\mathbf{i_C(t) = A_1 e^{-200t} + A_2 e^{-800t} \quad for\ t \ge 0^+}$$



Find:

a) The value of α, w_o, L, C, A_1, and A_2.
b) The expression for v(t), t ≥ 0.
c) The expression for $i_R(t)$, t ≥ 0.
d) The expression for $i_L(t)$, t ≥ 0.

Solution:

a) The value of α, w_o, L, C, A_1, and A_2.

The response given is for a RLC natural response $i_C(t) = A_1e^{-200t} + A_2e^{-800t}$ for t ≥ 0 and by comparing to possible response types:

$$Overdamped(\omega_0^2 < \alpha^2) \Rightarrow i_C(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

$$Critically\ Damped((\omega_0^2 = \alpha^2) \Rightarrow i_C(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t}$$

$$Underdamped(\omega_0^2 > \alpha^2) \Rightarrow i_C(t) = B_1e^{-\alpha t}\cos w_d t + B_2e^{-\alpha t}\sin w_d t$$

It can be concluded that it is overdamped response:

$i_C(t) = A_1e^{s_1t} + A_2e^{s_2t} = A_1e^{-200t} + A_2e^{-800t}$ ➔ s_1=-200; s_2=-800

Use the above information and definition of frequency and roots

$$\alpha = \frac{1}{2RC} = \frac{1}{500C}\ rad/\sec$$

$$w_0 = \sqrt{\frac{1}{LC}}\ rad/\sec$$

→ α=500 rad/sec; w_o=400 rad/sec; C=4 uF; L=1.56 H

$$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2} = -200\ rad/\sec$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - w_0^2} = -800\ rad/\sec$$

$i_C(0^+)$ = A1 + A2 = 0.045 A

KCL → $i_C + i_L + i_R = 0$ take a derivative →

$$\frac{di_c(0)}{dt} + \frac{di_L(0)}{dt} + \frac{di_R(0)}{dt} = 0$$

$$\frac{di_c(0)}{dt} = -\frac{di_L(0)}{dt} - \frac{di_R(0)}{dt} = \frac{v(0)}{L} - \frac{1}{R}\frac{i_c(0^+)}{C} \quad "hint"$$

$$\frac{di_c(0)}{dt} = \frac{15}{1.56} - \frac{1}{250}\frac{0.045}{4x10^{-6}} = -54.6A/s$$

$$\frac{di_c(t=0)}{dt} = 200A_1 + 800A_2 = 54.6$$

$$A_1 + A_2 = 0.045$$

$$Solve\ above\ two\ equations \Rightarrow A_1 = -31mA \quad \& \quad A_2 = 76mA \Rightarrow$$

$$i_c(t) = -31e^{-200t} + 76e^{-800t}mA \quad t \geq 0^+$$

b) The expression for v(t), t ≥ 0.

$$Given\ that\ the\ form\ will\ be \Rightarrow v_c(t) = A_3 e^{-200t} + A_4 e^{-800t} V \quad t \geq 0^+$$

$$v(0+) = A_3 + A_4 = 15$$

$$i_c(t) = C\frac{dv_c(t)}{dt} \Rightarrow \frac{dv_C(0)}{dt} = \frac{i_c(0)}{C} = \frac{0.045}{4x10^{-6}} = 11{,}250 \ \ V/s$$

$$-200A_3 - 800A_4 = 11{,}250$$

$$v(0+) = A_3 + A_4 = 15$$

$$Solve \Rightarrow A_3 = 38.75\,V; \ \ A_3 = -23.75\,V$$

$$v_c(t) = 38.75e^{-200t} - 23.75e^{-800t} V \quad t \geq 0^+$$

c) The expression for $i_R(t)$, t ≥ 0.

$$i_R(t) = \frac{v_c(t)}{R} = 155e^{-200t} - 95e^{-800t} mA \quad t \geq 0^+$$

d) The expression for $i_L(t)$, t ≥ 0.

$$i_L(t) = -i_R(t) - i_C(t) = -124e^{-200t} + 19e^{-800t} mA \quad t \geq 0^+$$

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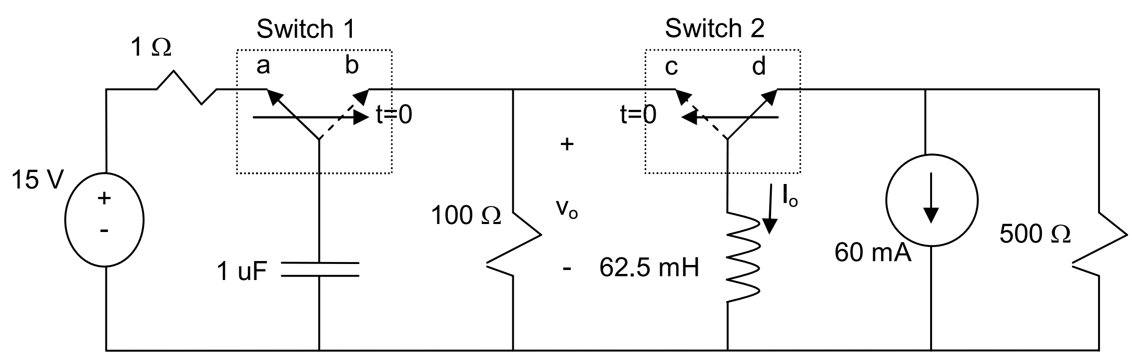
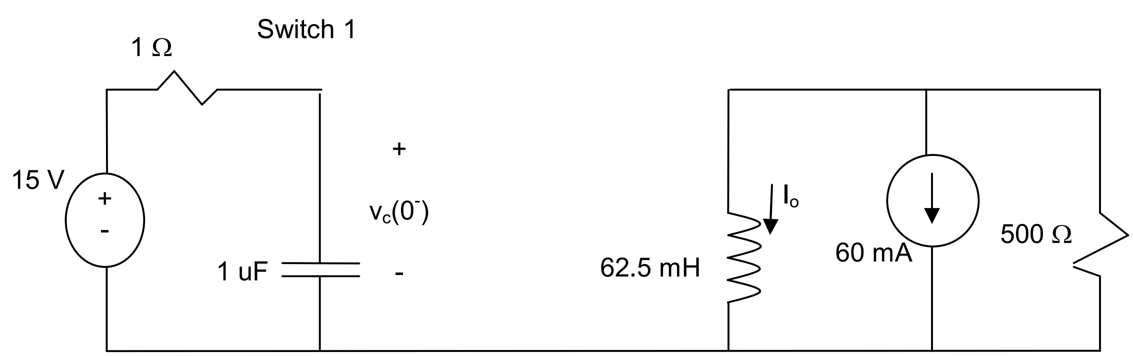
5) The two switches in the circuit seen below operate synchronously. When switch 1 is in position a, switch 2 is in position d. When switch 1 moves to position b, switch 2 moves to position c. Switch 1 had been in position a for a long time. At t=0, the switches move to their alternate positions. Find $v_o(t)$ for t ≥ 0.



Solution:

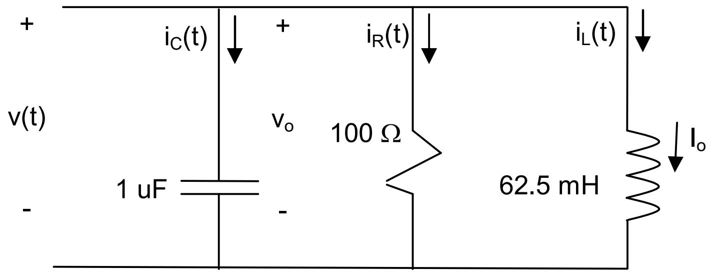
At t=0^- (before switches change position), the circuit is as follows:



Form RC and RL circuit analysis, it is understood that:

* Capacitor appears as an open after a long time → $V_c(0^-)$=15V
* Inductor appears as a short after a long time → Io = - 60 mA

At t=0^+ (after switches changes position), the circuit is as follows:



Find the natural response for a RLC parallel circuits.

$$\alpha = \frac{1}{2RC} = \frac{1}{2*100*1*10^{-6}} = 5000\ rad/\sec \Rightarrow \alpha^2 = 25*10^6$$

$$w_0{}^2 = \frac{1}{LC} = \frac{1}{63.5*10^{-3}*1*10^{-6}} = 16*10^6\ rad/\sec$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -5000 \pm 3000\ rad/\sec \Rightarrow s_1 = -2000\ rad/\sec; s_2 = -8000\ rad/\sec$$

The response is:

$$Overdamped\,(\omega_0^2 < \alpha^2) \Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{-2000t} + A_2 e^{-8000t}$$

Use the initial conditions to find A_1 and A_2:

$$ic(0) = -i_R(0) - i_L(0) = -\frac{v(0)}{R} - I_0 = -\frac{15}{100} - (-60) = -90mA$$

$$i_C(t) = C\frac{dv(t)}{dt} \Rightarrow \frac{dv(0)}{dt} = \frac{i_C(0)}{C} = \frac{-0.090}{10^{-6}} = -90{,}000$$

$$\frac{dv(t)}{dt} = -2000A_1 e^{-2000t} - 8000A_2 e^{-8000t}$$

$$-2000A_1 - 8000A_2 = -90{,}000$$

$$v(0) = A_1 + A_2 = 15$$

$$Solve \Rightarrow A_1 = 5V; \quad A_2 = 10V$$

$$v(t) = 5A_1 e^{-2000t} - 10A_2 e^{-8000t} \quad t \ge 0$$

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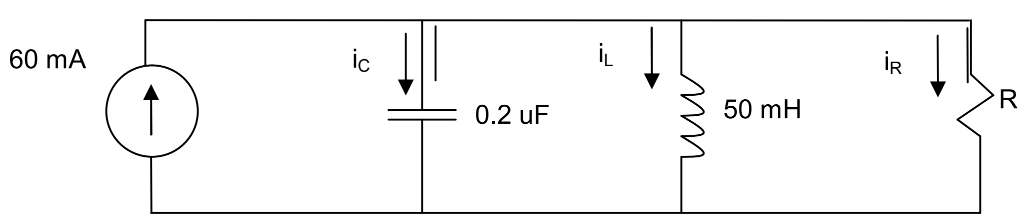
6) Assume that at the instant the 60 mA dc current source is applied to the following circuit where:
- **The initial current in the 50 mH inductor is -45 mA**
- **The initial voltage on the capacitor is 15 V (Positive at the upper terminal)**



a) Find the expression for $i_L(t)$ for t≥0 if R is equal 200 Ω.
b) Find the expression for $i_L(t)$ for t≥0 if R is equal 312.5 Ω.
c) Find the expression for $i_L(t)$ for t≥0 if R is equal 250 Ω.

Solution:

a) Find the expression for $i_L(t)$ for t≥0 if R is equal 200 Ω.

$$\alpha^2 = (\frac{1}{2RC})^2 = (\frac{1}{2*200*0.2*10^{-6}})^2 = (12500)^2$$

$$w_0{}^2 = \frac{1}{LC} = \frac{1}{50*10^{-3}*0.2*10^{-6}} = 10^8$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -12500 \pm 7500\ rad/\sec \Rightarrow s_1 = -5000\ rad/\sec; s_2 = -20{,}000\ rad/\sec$$

$$Overdamped(\omega_0^2 < \alpha^2) \Rightarrow i_L(t) = I_f + A_1 e^{s_1 t} + A_2 e^{s_2 t} = 0.06 + A_1 e^{-5000t} + A_2 e^{-20{,}000t}$$

Use initial condition to find A_1 and A_2

$$v(t) = L\frac{di_L(t)}{dt} \Rightarrow \frac{di_L(0)}{dt} = \frac{v(0)}{L} = \frac{15}{0.05} = 300$$

$$\frac{di_L(t)}{dt} = -5000 A_1 e^{-5000t} - 20{,}000 A_2 e^{-20{,}000t}$$

$$-5000A_1 - 20{,}000A_2 = 300$$

$$i_L(0) = 0.06 + A_1 + A_2 = -0.045$$

$$Solve \Rightarrow A_1 = -0.12A; \quad A_2 = 0.015A$$

$$i_L(t) = 0.06 - 0.12_1 e^{-2000t} + 0.015A_2 e^{-8000t}\ A \quad t \ge 0$$

b) Find the expression for $i_L(t)$ for t≥0 if R is equal 312.5 Ω.

$$\alpha^2 = (\frac{1}{2RC})^2 = (\frac{1}{2*312.5*0.2*10^{-6}})^2 = (8000)^2$$

$$w_0{}^2 = \frac{1}{LC} = \frac{1}{50*10^{-3}*0.2*10^{-6}} = 10^8$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -8000 \pm j6000\ rad/\sec$$

$$w_d = \sqrt{w_0^2 - \alpha^2} = 6000 rad/\sec$$

$$underdamped(\omega_0^2 > \alpha^2) \Rightarrow i_L(t) = I_f + B_1 e^{\alpha t}\cos w_d t + B_2 e^{\alpha t}\sin w_d t$$

$$i_L(t) = 0.06 + B_1 e^{-8000t}\cos 6000t + B_2 e^{-8000t}\sin 6000t$$

Use initial condition to find B_1 and B_2

$$v(t) = L\frac{di_L(t)}{dt} \Rightarrow \frac{di_L(0)}{dt} = \frac{v(0)}{L} = \frac{15}{0.05} = 300$$

$$\frac{di_L(t)}{dt} = -8000B_1 + 6{,}000B_2 = 300$$

$$i_L(0) = 0.06 + B_1 = -0.045$$

$$Solve \Rightarrow B_1 = -0.105A; \quad A_2 = -0.090A$$

$$i_L(t) = 0.06 - 0.105e^{-8000t}\cos 6000t - 0.090e^{-8000t}\sin 6000t\ A \quad t \ge 0$$

c) Find the expression for $i_L(t)$ for t≥0 if R is equal 250 Ω.

$$\alpha^2 = (\frac{1}{2RC})^2 = (\frac{1}{2*250*0.2*10^{-6}})^2 = (10^4)^2$$

$$w_0{}^2 = \frac{1}{LC} = \frac{1}{50*10^{-3}*0.2*10^{-6}} = 10^8$$

$$Critically\ damped(\omega_0^2 = \alpha^2) \Rightarrow i_L(t) = I_f + D_1 te^{-\alpha t} + D_2 e^{-\alpha t} = 0.06 + A_1 e^{-10,000t} + A_2 e^{-10,000t}$$

Use initial condition to find A_1 and A_2

$$v(t) = L\frac{di_L(t)}{dt} \Rightarrow \frac{di_L(0)}{dt} = \frac{v(0)}{L} = \frac{15}{0.05} = 300$$

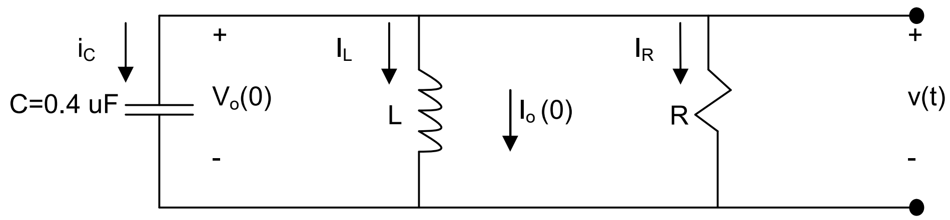
$$\frac{di_L(0)}{dt} = -10,000 D_2 + D_1 = 300$$

$$i_L(0) = 0.06 + D_2 = -0.045$$

$$Solve \Rightarrow D_1 = -750A; \quad D_2 = -0.105A$$

$$v(t) = 0.06 - 750e^{-10,000t} - 0.105e^{-10,000t} A \quad t \ge 0$$

E1)



The natural voltage response for the above circuit: is:

$v(t) = 100e^{-20,000t}[\cos(15,000t - 2\sin(15,000t)$ V for $t \ge 0$.

Find L, R, $V_o(0)$, and $I_o(0)$

Solution

Find L)

From value v(t) given then it is underdamped response →

$$v(t) = e^{-\alpha t}(B_1 \cos w_d t + B_2 \sin w_d t) = e^{-20,000t}(100\cos 15,000t - 200\sin 15,000t) \quad t \ge 0$$

$$\alpha = 20,000 \quad rad/\sec; \quad \omega_d = 15,000 \quad rad/\sec$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \Rightarrow 15,000 = \sqrt{\omega_0^2 - 20,000^2} \Rightarrow \omega_0^2 = 625x10^6$$

$$\omega_0 = \frac{1}{LC} = \frac{1}{L*0.4*10^{-6}} = 625x10^6 \Rightarrow L = 4mH$$

Find R)

$$\alpha = \frac{1}{2RC} \Rightarrow 20,000 = \frac{1}{2R*0.4*10^{-6}} \Rightarrow R = 62.5\Omega$$

Find V_o)

$$Vo = v(0) = 100 \quad V$$

Find I_o)

$$Io = i_L(0) = -i_R(0) - i_C(0)$$

$$i_R(0) = \frac{Vo}{R} = \frac{100}{62.5} = 1.6\ A$$

$$\frac{dv(t)}{dt} = 100\{e^{-20,000t}[-15,000\sin 15,000t - 30,000\cos 15,000t] - 20,000e^{-20,000t}[\cos 15,000t - 2\sin 15,000t]$$

$$i_C(0) = C\frac{dv}{dt}(0) = (0.4x10^{-6})\{100\{1(-30,000) - 20,000\} = -2\ A$$

$$Io = i_L(0) = -i_R(0) - i_C(0) = -1.6 - (-2) = 0.4\ A$$

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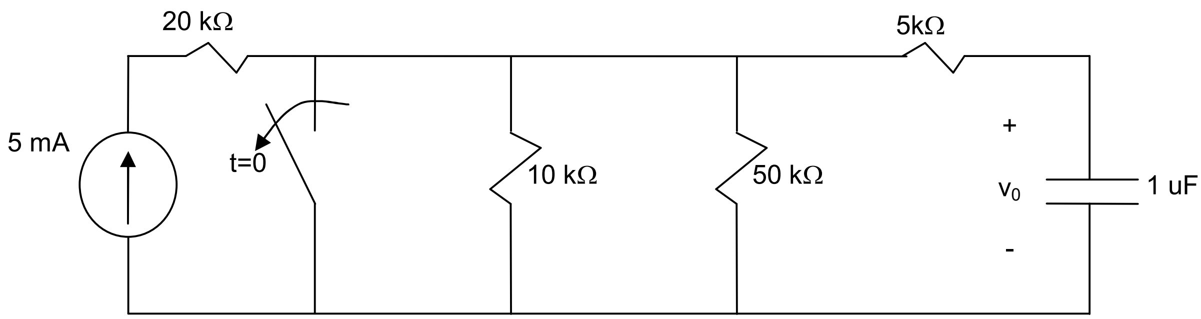
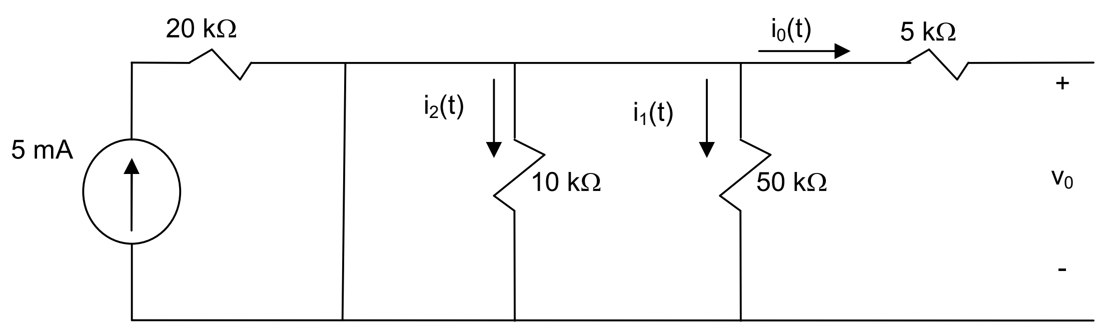
7. The switch in the following circuit has been closed a long time before opening at t = 0. For t ≥ 0+, find $v_0(t)$.



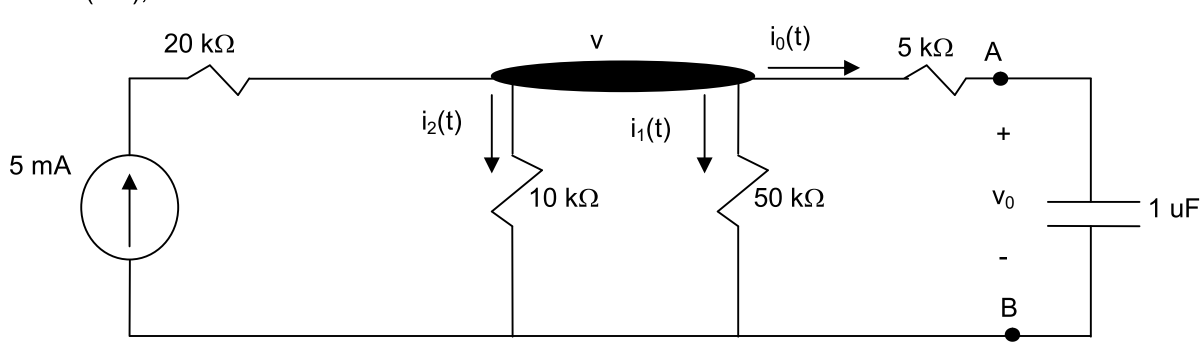
Solution:

After a long time at $t=0^-$, capacitor appears as open…



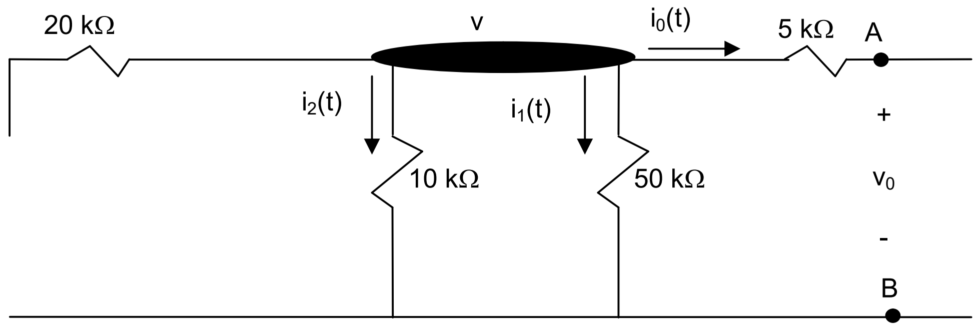
Short across the source then no power is deliver to the rest of the circuit →
$v_0(0^-) = 0$ V
$i_0(0^-) = i_1(0^-) = i_2(0^-) = 0$ A

After t=0 (t>0), circuit is redrawn as:

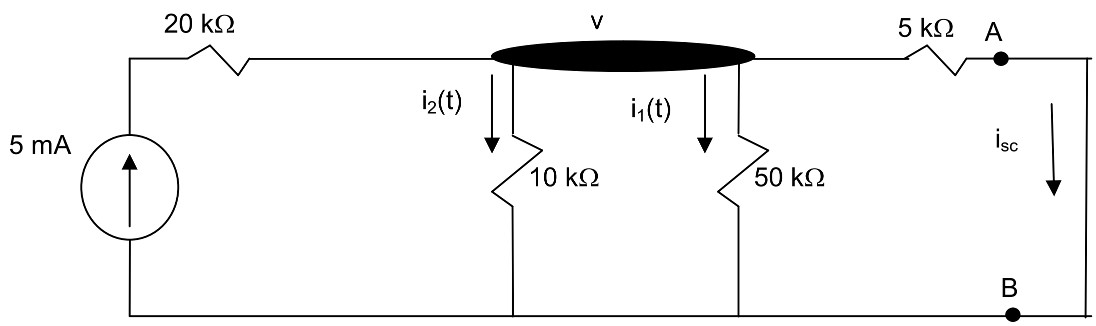


At t=0$^+$, Capacitor appears as a short → Vo(0+)=0 V.

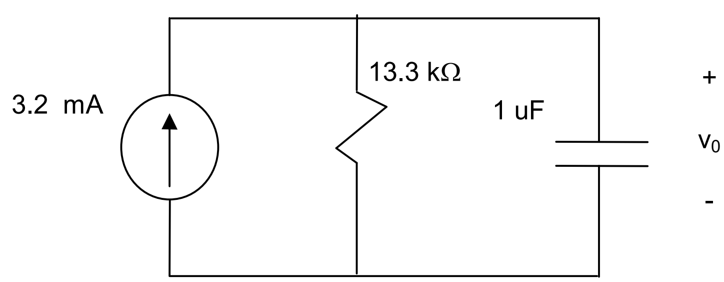
Find the Norton Equivalent at terminal AB→
Disable current source (open) to find Rth:



Rth = (10 || 50) + 5 = 13.3 kΩ
Find Isc:



KCL → -5 + v/10 + v/50 + v/5 = 0 → 16v = 250 → v(0$^+$) = 15.6 V → i_{sc}= 3.12 mA



Apply the step response for RC circuit $C\frac{dv}{dt}+\frac{v}{R}=I_s$

$$v(t)=I_sR+(V_0-I_sR)e^{-t/RC}\ for\quad t\geq 0$$

$$v_0(t)=0.0032*13{,}300+(0-0.0032*13{,}300)e^{-t/(13{,}300*10^{-6})}\ for\quad t\geq 0$$

$$v_0(t)=42.6-42.6e^{-75.2t}V\quad for\quad t\geq 0$$