

Fundamentals of
Electrical Circuits

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Background and Acknowledgements

This material is intended for the first course sequence in Electrical Engineering focused on Electrical Circuit Analysis and Design. The content is derived from the author's educational, engineering and management career, and teaching experience. Additionally, the following resources have informed the development of content and format:

- Katz, R. Contemporary Logic Design. (2005) Pearson.
- Wakerly, I. Digital Design. (2001) Prentice Hall.
- Sandige, R. Digital Design Essentials. (2002) Prentice Hall.
- Nilsson, J. Electrical Circuits. (2004) Pearson.
- MathWorks. MATLAB Reference Material Version R2000a. (2007) MathWorks

I would like to extend special thanks to the many students and colleagues for their contributions in making this material a more effective learning tool.

Further, I would invite the reader to forward corrections, additional topics, examples and problems to me for future updates.

Thanks,

Izad Khormaei

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Chapter 1. Introduction

Key Concepts and Overview

- ❖ Overview of Electrical Engineering
- ❖ Problem Solving
- ❖ Systems of Units
- ❖ Charge, Current & Voltage
- ❖ Circuit Model
- ❖ Power and Energy
- ❖ Additional Resources

1.1. Overview of Electrical Engineering

This course builds each new concept on material covered earlier in the course. Therefore, it is important to master each concept before moving on to the next section. Additionally, the materials presented here are the foundations that all future electrical engineering education is built on.

Engineers are creative problem solvers who use their knowledge of mathematics and science to solve problems that add value to society. Specifically, electrical engineering is a profession concerned with systems that produce, transmit, control and measure electrical signals. Samplings of electrical engineering fields are presented below:

- Automation & Robotics
- Bio/Medical
- Communication
- Computer
- Control
- Manufacturing
- Power Generation and Distribution
- Semiconductor
- ...

The material covered here is typically taught in second and third year of electrical and computer engineering degree programs. Follow on courses expand on the topics covered here with emphasis on one of the specialized areas in more depth. Below are a sampling of specialization areas in electrical and computer engineering:

- Communication
- Computers
- Electro Magnetic and fields
- Electronics and Digital Circuits
- Networking
- Power and Energy Systems
- Robotics
- Semiconductors and Devices
- Sensors
- Control Systems
- Bio/Medical
- Photonics
- Machine Learning
- Gaming
- Motion Systems
- Micro Electro Mechanical (MEM) systems
- Nano Technology

1.2. Problem Solving

An engineer's major responsibility is to solve problems through the creative application of science and mathematics knowledge. Therefore, as engineers, it is important to develop strong problem-solving skills.

It is recommended that the following steps be used in solving assigned problems and labs relating to this material:

1. Identify the information provided and list them under the title "Given"
2. Identify the information to be derived and list them under the heading "Find"
3. Use circuit diagrams and visual models to clarify the problem
4. Develop possible models and methods to solve the problem
5. Pick the optimum model or method to solve the problem
6. Solve the problem
7. Test the validity of your solution

Even though the above problem-solving steps are listed sequentially, the process of problem solving is an iterative process. This fact reinforces the need to use a systematic approach that is repeatable and easily understandable in each subsequent iteration.

❖ End of Chapter Problems

At the end of each chapter there are a number of problems designed to deepen our understanding of the material covered. It is very important that students solve all the end of chapter problems.

1.3. Systems of Units

The International System of Units (SI) is the formal name for the metric system. It is the most widely used system of units by professionals engaged in engineering, commerce and science.

Measurements may be categorized as either Measured or Derived:

- **SI Measured (or Base) Units**
Measured units are also called SI Base units. The seven measured units are defined by actual measurement of a specific physical object or phenomenon.
 - 1) Length, Meter (m)
 - 2) Mass, Kilogram (kg)
 - 3) Time, Seconds (s)
 - 4) Electric Current, Amp (A)
 - 5) Thermodynamic Temperature, Kelvin (K)
 - 6) Amount of substance, Mole (mol)
 - 7) Luminous intensity, Candle (cd)
- **SI Derived Units**
These units are derived from the Base units which explain why they are called the SI derived units:
 - 1) Frequency, Hertz (H) = cycle/s
 - 2) Force, Newton (N) = $\text{kg} \cdot \text{m/s}^2$
 - 3) Energy, Joule (J) = $\text{N} \cdot \text{m}$
 - 4) Power, watt (W) = J/s
 - 5) Electric Charge, Coulomb (C) = $\text{A} \cdot \text{s}$
 - 6) Electric Potential or Voltage, Volt (V) = J/C
 - 7) Electric Resistance, Ohm (Ω) = V/A
 - 8) Electric Conductance, Siemens (S) = A/V
 - 9) Electric Capacitance, Farad (F) = C/V
 - 10) Magnetic flux, Weber (Wb) = $\text{V} \cdot \text{s}$
 - 11) Inductance, Henry (H) = Wb/A

Electrical measurements may result in very small or very large numbers. In order to simplify the communication of data in such a wide range, powers of 10 are used to avoid long lists of zeros. When powers of 10 are in multiples of three, it is referred as Engineering Notation (for example 4×10^6). Furthermore, specific powers of ten have been named for improved communication. Below are some of the most common named powers of 10 used in engineering:

Prefix	Symbol	Powers of 10
Atto	a	10^{-18}
Femto	f	10^{-15}
Pico	p	10^{-12}
Nano	n	10^{-9}
Micro	μ	10^{-6}
Milli	m	10^{-3}
Kilo	K	10^3
Mega	M	10^6
Giga	G	10^9
Tera	T	10^{12}
Peta	P	10^{15}
Exa	E	10^{18}

Using the prefixes defined within the above table the value 3,000,000 Ω may be restated as 3 M Ω and 0.004 F may be restated as 4 mF.

❖ Example of Unit Conversions

If a signal travels in a cable at 80% of the speed of light, what length of cable in inches does it travel in 1 nSec.

Hint: Speed of light = 3×10^8 m/s

➤ Solution

1) Given:

Signal Speed : 80% of Speed of light

Travel time: 1 nSec.

Speed of light: 3×10^8 m/s

2) Find:

lc: Length of Cable

3) Method

Use the SI units and cancellation technique

4) Answer

$$lc = 3 \times 10^8 \text{ m/s} * 80/100 * 1 \text{ Sec}/10^9 \text{ nSec} * 100 \text{ cm/m} * 1 \text{ in}/2.54 \text{ cm} = \underline{9.45 \text{ inches}}$$

1.4. Charge, Current and Voltage

❖ Electric Charge

A charge may be either electric, color or magnetic charge. We will be focusing on electric charge here. Electric charge is the basic property of some subatomic particles. Electrons by convention have a charge of -1 and, similarly, protons have the opposite Charge of +1. In general, electric charge is bipolar. Meaning that electrical effects are described as having positive and negative charges.

The SI unit for electric charge is the Coulomb, which represents approximately 6.24×10^{18} elementary charges. Elementary Charge is the charge on a single electron or proton. In other word electric charge exists in discrete quantities (elementary charges) which are integral multiples of the electric charge, $1.6022 \times 10^{-19} \text{ C}$.

❖ Electric Current and Voltage

Electrical effects are attributed to both the separation of charge (Voltage) and charges in motion (Current).

➤ Current

Amount of current is measured in Amperes, which at its simplest is the change of charge q over time and can be expressed as $i = \frac{dq}{dt}$ where:

i = current in Amperes
 q = charge in Coulombs
 t = time in Seconds

➤ Voltage

Amount of potential is measured in Volts, and potential represents the amount of energy required to move electric charge from point a to b. Therefore, potential is expressed as $v = \frac{de}{dq}$ where:

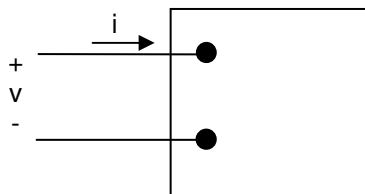
v = potential in Volts
 e = energy in Joules
 q = charge in Coulombs

Voltage, V , is the difference in potentials between two points in the circuits.

❖ Example – Charge & Current

For the following the circuit, calculate the total charge in micro-coulombs entering the upper terminal.

$$\begin{aligned} i &= 0 & t < 0; \\ i &= 20e^{-5000t} \text{ A} & t \geq 0 \end{aligned}$$



➤ Solution:

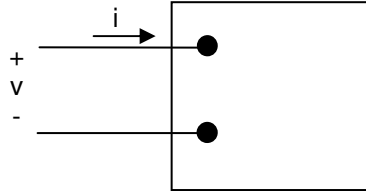
$$i = \frac{dq}{dt} \Rightarrow q = \int_{-\infty}^{+\infty} i dt = 0 + \int_0^{+\infty} 20e^{-5000t} dt = \frac{20}{-5000} (e^{-5000t}) \Big|_0^{+\infty} = \frac{20}{-5000} (0 - 1) \text{ Coulombs}$$

$$q = 4000 \mu\text{C}$$

❖ Example - Charge & Current

For the specified charge entering the upper terminal of the circuit below, find the maximum value of the current entering the terminal if $a = 0.03679 \text{ s}^{-1}$.

$$q = 1/a^2 - (t/a + 1/a^2)e^{-at} \text{ C}$$



➤ Solution:

$$i = \frac{dq}{dt} = 0 - (1/a + 0)e^{-at} - (t/a + 1/a^2)(-a)e^{-at} = te^{-at}$$

$$i \text{ is max/min} \Rightarrow \frac{di}{dt} = 0 \Rightarrow \frac{di}{dt} = e^{-at} - ate^{-at} = (1 - at)e^{-at} = 0$$

$$i_{\text{max}} \text{ when } t = 1/a \text{ \& we have } a = 0.03679 \text{ s}^{-1} \Rightarrow$$

$$i_{\text{max}} = te^{-at} = (1/a)e^{-a(1/a)} = (1/0.03679)e^{-1} = 10 \text{ A}$$

❖ Example - Charge & Current

Given the energy equation " $e = 200e^{-100t}q$ ", find the equation for voltage, v .

➤ Solution:

Sidebar - Useful Derivatives

$$\frac{d(e^{at})}{dt} = ae^{at}$$

$$\frac{d(\cos(at))}{dt} = -a \sin(at)$$

$$\frac{d(\sin(at))}{dt} = a \cos(at)$$

$$\frac{d(f(t)g(t))}{dt} = f'(t)g(t) + f(t)g'(t)$$

1.5. Circuit Model

An Electrical Circuit is a mathematical model that approximates the behavior of an actual electrical system. Ideal Circuit Theory is a special case of electromagnetic field theory. The three assumptions that allow the use of the Ideal Circuit Theory are:

- Electrical effects propagate instantly through the whole system
The systems that are small enough relative to signal wavelength for this assumption to be valid are called lumped systems. If the wavelength of the signal is more than 10 times the physical dimension of the system then we have a lumped system. As a reminder, the formula for calculating wavelength is shown below:

$$\text{Wavelength } \lambda = (\text{speed of light, } 3 \times 10^8 \text{ m/s}) / (\text{signal frequency, } f \text{ Hertz})$$

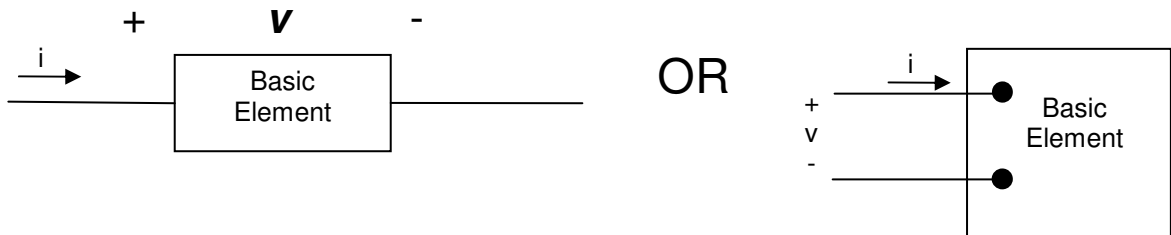
Student Exercise: If signal frequency is 20 MHz, what is the maximum size for a circuit that can be classified as a lumped circuit?

- Net charge of every component in the system is zero
This assumption states that none of the components collect net charges or lose net charges. In other word, the charge that enters a component, must leave the component.
- No magnetic couplings
No magnetic coupling means that charges can only flow through physically connected components by a conducting material (typically wires). It follows that, with this assumption, charges cannot leave the circuit through air.

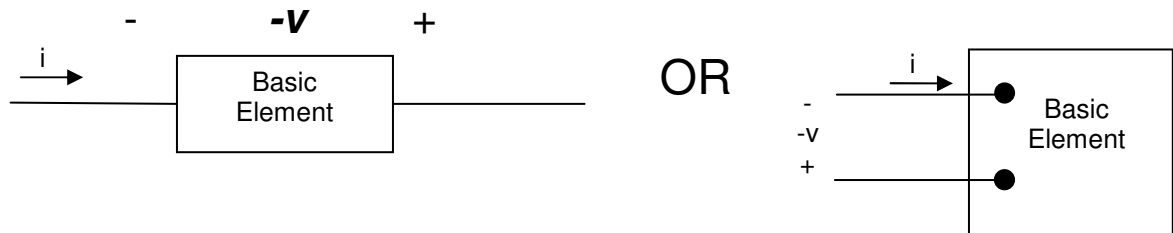
- ❖ Circuit analysis uses idealized basic circuit elements and ideal circuit model (a mathematical model) in this text. Earlier the characteristic ideal circuit model was defined and the following three attributes characterize an idealized basic circuit element:

- It has only two terminals (terminals are the points of connection to other elements).
- It can be fully described in terms of the relationship between current and voltage.
- It cannot be subdivided into other elements.

Electrical engineering typically uses the passive convention which assumes the current enters at the positive voltage terminal. Two layouts of basic elements are shown below using passive convention:



You may use a negative sign to reverse the voltage polarity or reverse current direction. For example the above model may be redrawn as:



❖ Ideal Basic Elements

The rest of this text uses the following four basic ideal elements to model electrical systems. As we progress through the material, one-by-one the simplifying assumptions will be removed to better approximate the actual electrical characteristic of a system.

The four elements are listed below:

- Power source
- Resistor
- Capacitor
- Inductor

Each of these elements will be defined in latter chapters.

1.6. Power and Energy

Power and energy are related in that power is the change of energy over time as shown below:

$$P = \frac{de}{dt} \text{ where:}$$

P = the power in watts
e = the energy in joules
t = the time in seconds

Power can also be described in terms of current, i, and voltage, v, as shown below:

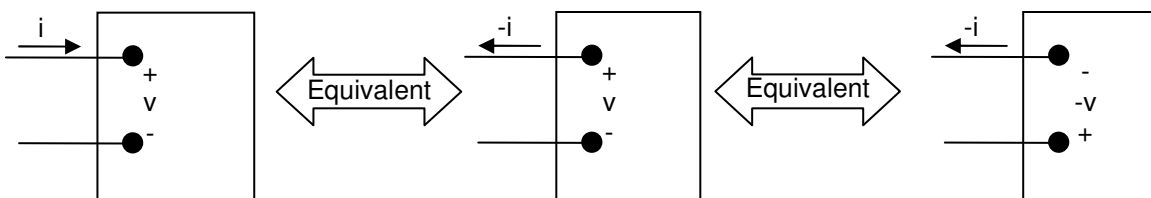
$$P = \frac{de}{dt} = \frac{de}{dt} * \frac{dq}{dq} = \frac{de}{dq} * \frac{dq}{dt} = v * i$$

Note: q represents the charge and (de / dq) is equal to voltage.

$P=V*I$ is the most common equation used for deriving power.

❖ Passive Sign Convention

As mentioned earlier, passive sign convention is the most common circuit analysis convention. It assumes that current following into the positive terminal results in positive power, otherwise the power is negative.



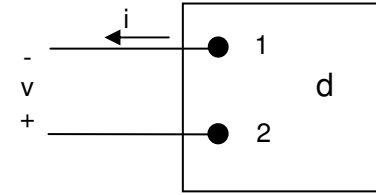
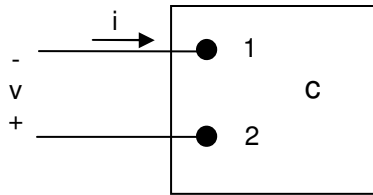
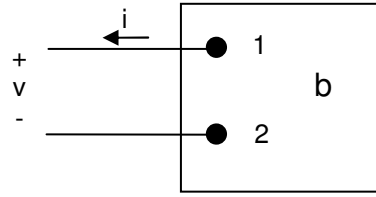
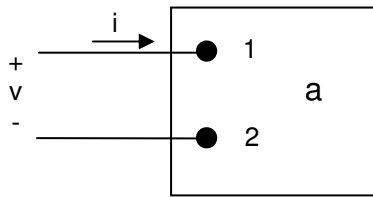
The above diagram shows that a negative sign is the same as changing current direction or swapping the polarity of voltage.

The following statements are valid for Passive Sign Convention:

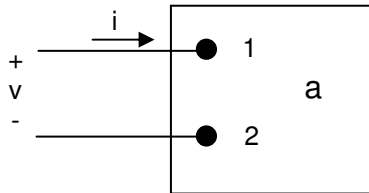
- 1) Positive power ($P > 0$) indicates that the element is consuming power.
- 2) Negative power ($P < 0$) indicates that the element is generating power.

❖ Example – Power and passive convention

For the following four elements where 20V voltage drops from terminal 2 to terminal 1 and a current of 4 A enters terminal 2, specify the value of V, I and power for each of the circuits. Additionally, state whether the element is absorbing or delivering power:



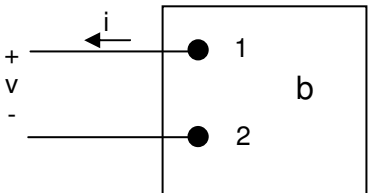
➤ Solution:



$$V = -20 \text{ V}$$

$$I = -4 \text{ A}$$

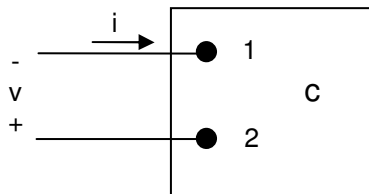
$$P = vi = 80 \text{ W (P is positive so it is absorbing power)}$$



$$V = -20 \text{ V}$$

$$I = 4 \text{ A}$$

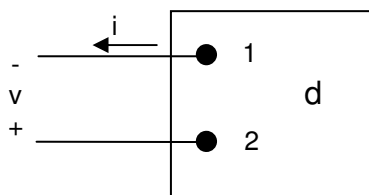
$$P = v(-i) = -80 \text{ W (P is positive so it is delivering power)}$$



$$V = 20 \text{ V}$$

$$I = -4 \text{ A}$$

$$P = (-v)i = -80 \text{ W (P is positive so it is delivering power)}$$



$$V = 20 \text{ V}$$

$$I = 4 \text{ A}$$

$$P = vi = 80 \text{ W (P is positive so it is absorbing power)}$$

❖ Example – Energy Calculation

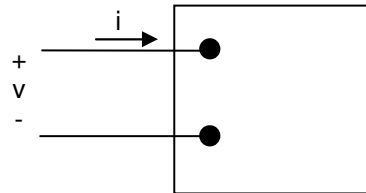
For the following circuit calculate the total energy in Joules delivered to the circuit.

$$i=0 \quad t<0;$$

$$i=20e^{-5000t} \text{ A} \quad t\geq 0$$

$$v=0 \quad t<0;$$

$$v=10e^{-5000t} \text{ KV} \quad t\geq 0$$



➤ Solution

$$P = v * i$$

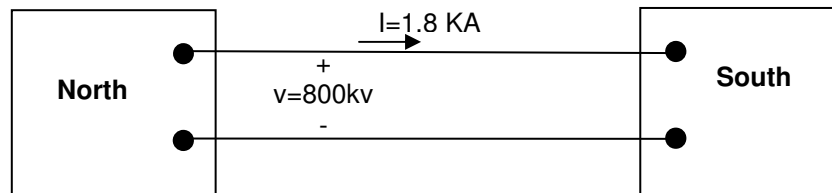
$$P = 0 \quad t < 0$$

$$P = (20e^{-5000t}) * (10000e^{-5000t}) = 200,000e^{-10,000t}$$

$$P = + \frac{dw}{dt} \Rightarrow w = \int_{-\infty}^{+\infty} P dt = \int_0^{+\infty} 200,000e^{-10,000t} dt = -20e^{-10,000t} \Big|_0^{\infty} = 20 \text{ Joules}$$

❖ Example – Multi-element circuit

A high-voltage direct-current (dc) transmission line between North and South is operating at 800 kV and carrying 1800 A as depicted in the diagram. Calculate the power in MegaWatts (MW) at the North end of the line and state the direction of power flow.



➤ Solution:

1) Option 1 – using South as a reference

Using passive sign convention (current to positive terminal) we have

$$i = +1.8 \text{ KA} \text{ \& } v = +800 \text{ KV where } P = v * i = 1.8 * 800 = 1440 \text{ MW}$$

(sign is positive so the power is being delivered to South which means it is being generated by North)

2) Option 2 – using North as a reference

using passive sign convention (current to positive terminal) we have

$$i = -1.8 \text{ KA} \text{ \& } v = +800 \text{ KV where } P = v * i = -1.8 * 800 = -1440 \text{ MW}$$

(sign is negative so the power is being generated by North and is being consumed by South)

❖ Example – Power Unit Variation

How long will a \$1 of power operate a 13 W light bulb?

Assuming: Power cost at \$0.08 per Kilo-Watt-Hour (KWH).

➤ Solution:

$$\text{\$1} * \frac{1\text{KWH}}{\text{\$0.08}} * \frac{1000\text{ W}}{1\text{ KW}} * \frac{1\text{bulb}}{13\text{W}} = 961.5\text{ hours}$$

1.7. Summary

1.8. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 1.

1.9. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 2. Basic laws

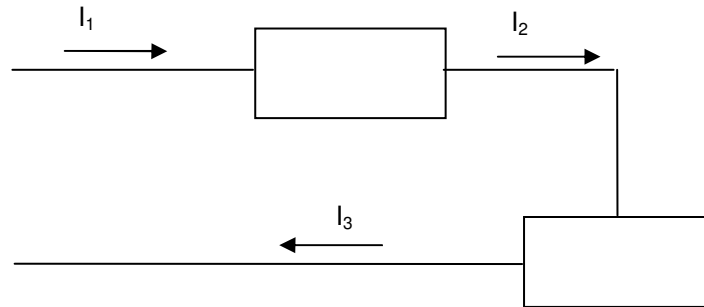
Key Concepts and Overview

- ❖ Ideal Voltage and Current Sources
- ❖ Ohm's Law
- ❖ Circuit Analysis
- ❖ Kirchhoff's Laws
- ❖ Additional Resources

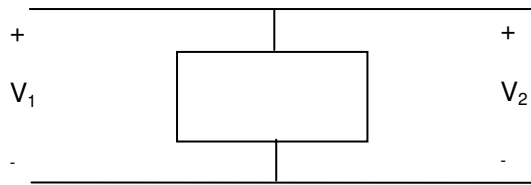
2.1. Ideal Voltage & Current Sources

Power source is a general term for voltage or current source. The power source is important to the analysis of electrical circuits since it provides the electrical excitation or energy required to operate the circuits. The ideal power source also adheres to the three ideal circuit characteristics which are restated below in terms of current, voltage and power:

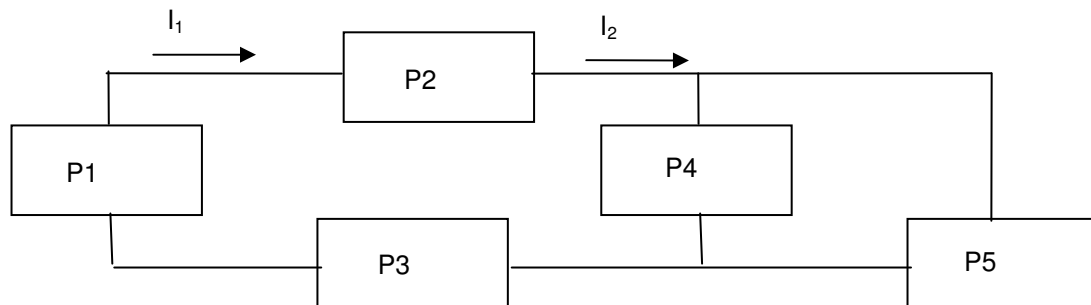
- 1) Current measured on various parts of connecting lines (no branches) is the same. For example, I_1 , I_2 and I_3 have the same value in the following Circuit.



- 2) Voltages across two terminals of an element are the same regardless which side of the terminal the measurement is made. For example, V_1 and V_2 have the same value in the following circuit.



- 3) Sum of all the powers consumed and generated by elements in a circuit equals zero, or stated another way "Generated power = Consumed power". For example in the following circuit, the equation $\{P1 + P2 + P3 + P4 + P5 = 0\}$ is true.

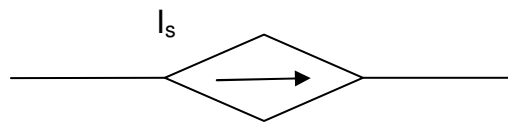


Power sources can be categorized based on the following characteristics:

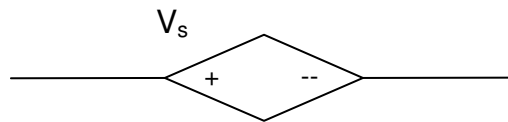
- **Current or Voltage Source**
A power source is called a current source if the current output is constant, and a voltage source if the voltage output is constant.
- **Dependent or Independent Source**
If the source output function does not depend on the rest of circuit, it is called an independent or uncontrolled source (represented by a circular symbol). When the source output depends on a parameter in another part of the circuit, it is called a dependent or controlled source (represented by a diamond symbol).

Using the above two characteristics (dependent/independent and current/voltage), the power sources may be classified as one of the following four types:

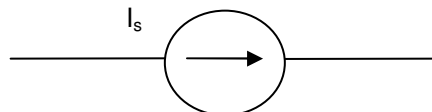
- 1) **Dependent Current Source**
Ideal Dependent Current Source, $I_s = u \cdot I_1$ or $I_s = u \cdot V_1$, where I_1 and V_1 are parameters in other parts of the circuit and u is a constant.



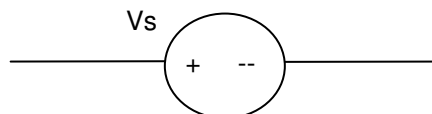
- 2) **Dependent Voltage Source**
Ideal Dependent Voltage Source, $V_s = u \cdot V_1$ or $V_s = u \cdot I_1$, where I_1 and V_1 are parameters in other parts of the circuit and u is a constant.



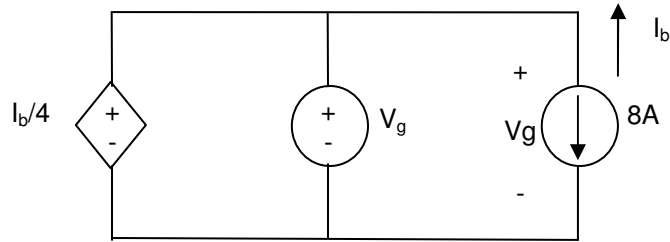
- 3) **Independent Current Source**
Ideal Independent Current Source, $I_s = 5 \text{ A}$ or $I_s = f(t)$, but does not depend on the rest of the circuit.



- 4) **Independent Voltage Source**
Ideal Independent Voltage Source, $v_s = 3 \text{ V}$ or $v_s = f(t)$, but does not depend on the rest of the circuit.



- ❖ **Example- Power Source Types**
For the circuit shown here:



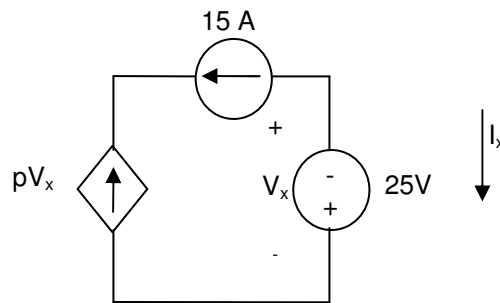
- What value of v_g is required in order for the interconnection to be valid?
- For the value of v_g from part a, find the power associated with the 8A independent source.

➤ Solution:

- Since this is an ideal circuit then $V_g = I_b/4$ (using ideal circuit Characteristic #2)
We have $I_b = -8A$ (opposite direction) $\rightarrow V_g = -2$

- $P_{8A \text{ Source}} = V \cdot I$
Passive sign convention (current flows to + terminal) $\rightarrow I = 8A$
 $P_{8A \text{ Source}} = V_g \cdot I = (-2) \cdot (8) = -16 \text{ W}$ "generating power since $P < 0$ "

❖ Examples - For the Circuit Shown



- What value of p is required in order for the interconnection to be valid?
- For the value of p calculated in last part, find the power associated with the 25V source.

➤ Solution

- ideal circuit $\rightarrow pV_x = -15A$ (using ideal circuit Characteristic #1)
We have $V_x = -25V \rightarrow p = 0.6$

- $P_{25\text{volt Source}} = V_x \cdot I_x$
Passive sign convention (current flows to + terminal) and we have $I_x = -15A$
 $P_{25\text{volt Source}} = V_x \cdot I_x = (-25) \cdot (-15) = 375$ "Consumed power since $P > 0$ "

2.2. Ohm's Law

Ohm's Law is one of the most fundamental laws of Electrical Circuits. Ohm's Law applies to both active and passive devices. An active device is one that is capable of generating power for example a current or voltage source. On the other hand, a passive device does not generate power, but it may use or store power. Examples of passive devices include resistors (R), Capacitors (C) and Inductors (L), which will be characterized later.

For now, let's focus on resistors and the definition of Ohm's law.

❖ Electrical Resistor

Resistor refers to materials that impede (resist) the flow of current or electric charges. Resistance is a measure of resistivity and is expressed in units of Ohms with symbol Ω .

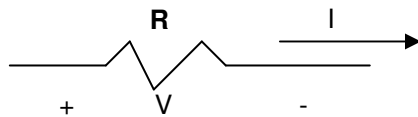
The flow of charges (current) through the resistor converts electrical energy to thermal energy (heat). Based on the application, the generated heat may be considered desirable or undesirable:

- Undesirable applications include electrical transmission lines and digital devices.
- Desirable applications include heaters, toasters and ovens.

❖ Resistance in a basic ideal element is defined in terms of current and voltage. This relationship was discovered by George Simon Ohm.

- Ohms Law may be stated as $V = I \cdot R$, $R = V/I$ or $I = V/R$, where:
 - R is in Ohms, Ω (Resistance of the element)
 - V is Volts, V (Voltage drop across the element)
 - I is in Amps, A (Current through the element)

Below is a resistor representation using passive sign convention (recommended for circuit analysis). Note that current always flows from the + to – terminal through the resistor (passive component):



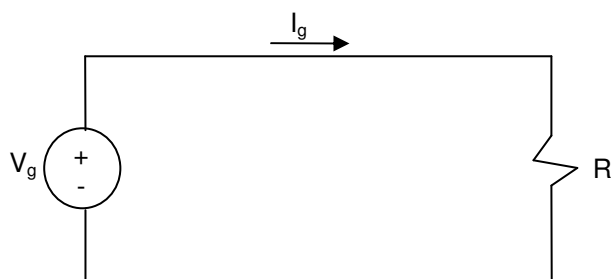
❖ Conductance is the inverse of resistance ($G = 1/R$) and is expressed in units of Siemens(S). In older texts, units may be referred to as mhos with inverted Ω symbol. Conductance is a measure of the ease by which charges follow through the material (lack of resistance).

❖ Power Formulas

Three ways to relate power with current, voltage and resistance:

- $P = V \cdot I$
- Use equation $I = V/R \rightarrow P = V^2 / R$
- Use equation $V = IR \rightarrow P = I^2 \cdot R$

❖ Example- for the following circuit Diagram



Find the value of R and the power absorbed by the resistor If $V_g = 1 \text{ KV}$ and $I_g = 5 \text{ mA}$.

➤ Solution

- $R = V_g / I_g = 1000 / 0.005 = 200 \text{ K}\Omega$
 $P_r = I_g^2 * R = (.005)^2 * (200,000) = 5 \text{ W}$

❖ Example – For the following circuit, when $I_g = 0.5 \text{ A}$ and $G = 50 \text{ mS}$ (Siemens):



Find V_g and the power delivered by the current source.

➤ Solution

- $V_g = I_g / G = 0.5 / 0.05 = 10 \text{ V}$
 $P_G = I_g^2 / G = (.5)^2 / (0.05) = 5 \text{ W}$

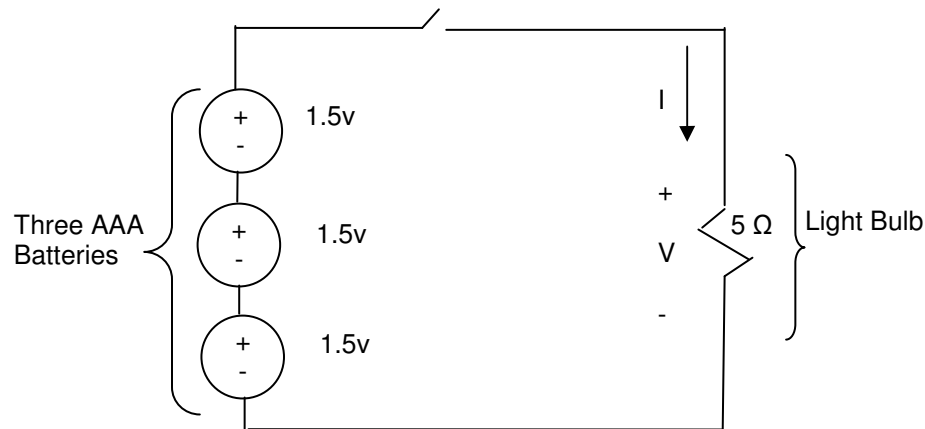
2.3. Circuit Modeling and Analysis

So far, the basic components of a circuit (power source and resistor), the Ideal Circuit Model and Ohms Law have been introduced. With the knowledge gained so far, we are able to model electrical systems as circuit models and analyze their electrical properties. Below are two examples of electrical circuits:

❖ Flash Light as an ideal circuit

A flash light is a an electrical circuit with the following components:

- A light bulb can be modeled as a resistor that converts electrical energy to thermal energy that causes radiation in the visible range.
- Batteries as the power source (non-ideal since the voltage and current drops over the life)
- On/Off Switch



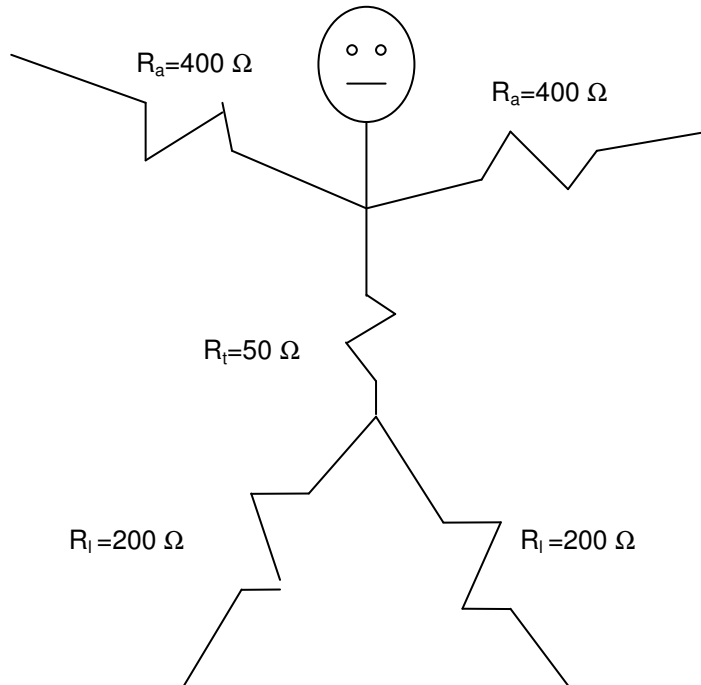
Find the value of current and power assuming an ideal circuit:

$$I = V/R = (1.5+1.5+1.5)/5 = 0.9 \text{ A}$$

$$P = I^2 * R = (0.9)^2 * (5) = 4.05 \text{ W}$$

❖ A typical person as an ideal electrical circuit

Humans can be modeled as an electrical circuit in order to understand the impact of electricity. Here is an example of a human modeled in terms of resistance:



Although burns due to high voltage are a problem, they rarely cause death.

Current effect on the human nervous system could be severe. For example if the nerves being affected are those that control the heart.

Here are current levels and the human reactions:

<u>Current</u>	<u>Physiological Response</u>
3-5 mA	Barely Perceptible
35-50 mA	Extreme Pain
50 – 70 mA	Muscle Paralysis
> 500 mA	Heart Stoppage

This model can be used to understand the level of current, voltage and power applied during an electrical shock in case of an accident or as part of a medical treatment.

2.4. Kirchhoff's Laws

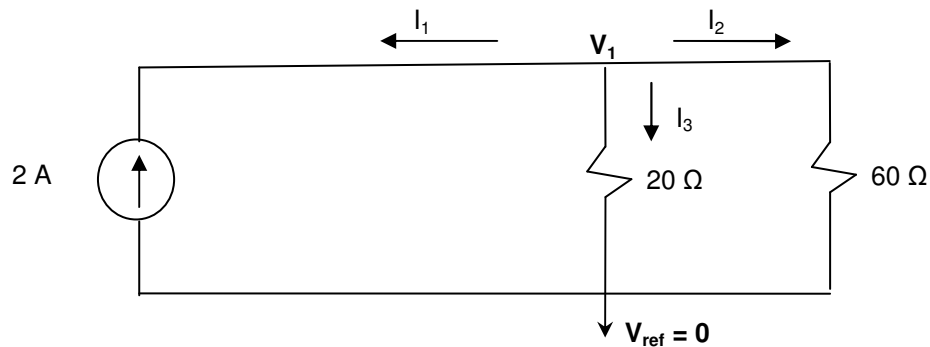
Solving a circuit problem means finding current, voltage, power and resistance. Kirchhoff's laws are important to our ability to solve circuit problems. Two variations of Kirchhoff's law are discussed in this section:

- Kirchhoff's Current Law
- Kirchhoff's Voltage Law

❖ Kirchhoff's Current Law (KCL) Definitions

- An essential node is a point on the circuit where more than 2 circuit elements connect.
- The algebraic sum of all the currents at any node in a circuit equals zero. Stated another way, any charge flowing into the node must exit (no storage capacity at the node).
- It is recommended to assume current flowing out of the node as the positive direction. Although, you may assume current flowing in as positive, as long as you are consistent!
- In any circuit with n node, we set one node as reference ($V_{ref}=0V$) and write KCL equations for remaining $(n-1)$ nodes.
- It is important to note that not all equations generated by Kirchhoff's laws are independent.

- Example - Kirchhoff's Current Law (KCL)
Apply Kirchhoff's Current law to find the value of I_2 .

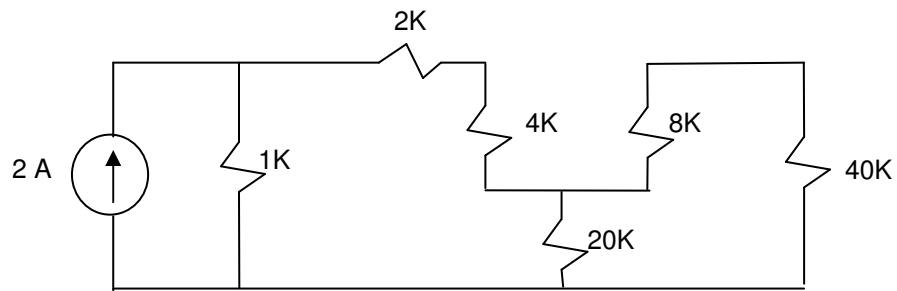


- Solution
 - (1) Identify the essential nodes and set one as reference (V_1 , V_{ref})
 - (2) For the essential node show currents leaving the node (I_1 , I_2 , I_3)
 - (3) Write the KCL equation for essential node(s) other than reference node
 $I_1 + I_2 + I_3 = 0$.
 - (4) Use Ohm's law and known values to rewrite the equations in terms of node voltage (reducing number of variables):
 - (a) I_1 is given as $-2A$
 - (b) $I_2 = V_1/60 A$
 - (c) $I_3 = V_1/20 A$

$$-2 + V_1/20 + V_1/60 = 0 \rightarrow V_1 = 30 V$$

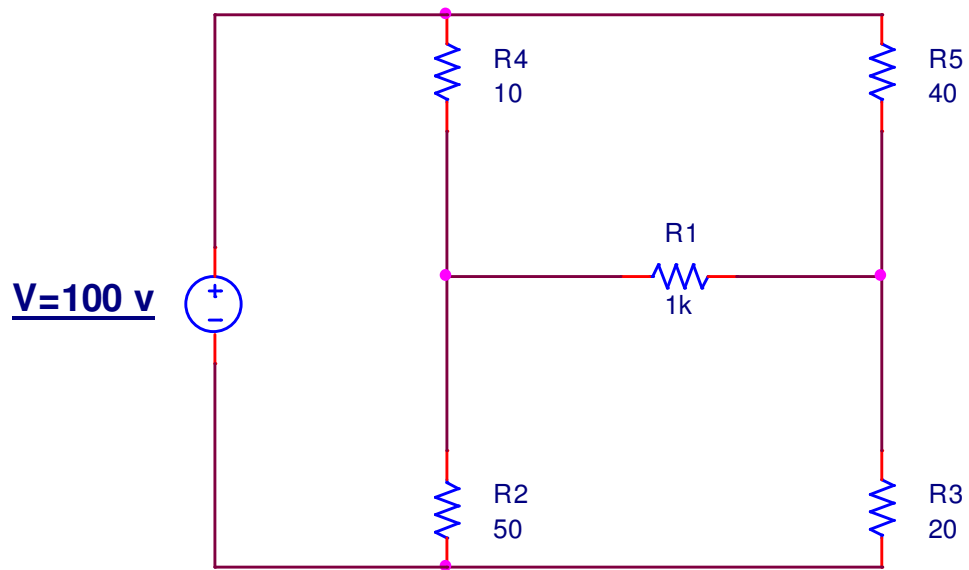
$$I_2 = 30/60 = 1/2 A$$

- Example - Kirchhoff's Current Law (KCL)
Find the voltage across 1K resistor using KCL.



- Solution:

- Example - Kirchhoff's Current Law (KCL)
Find the current across R_1 using KCL.



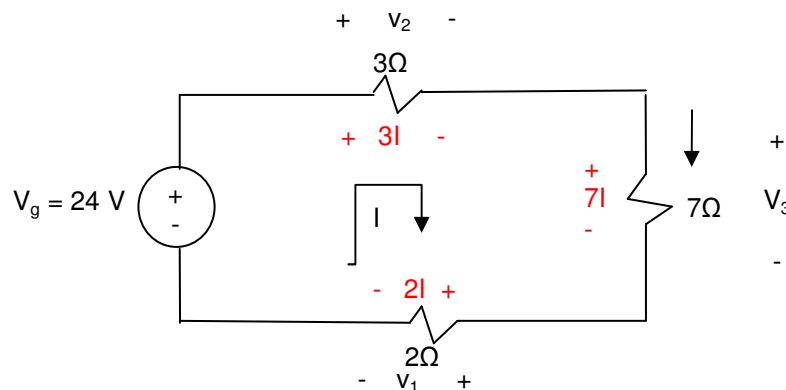
- Solution (Ans: 50 mA):

❖ Kirchhoff's Voltage Law (KVL) Definitions

- A closed path or loop is a path in a circuit through selected basic circuits elements that return to the original node without passing through any intermediate node more than once.
- The algebraic sum of all the voltages around any closed path in a circuit equals zero. If the + terminal is encountered first, the voltage is positive; otherwise, the voltage is negative.
- Pick either clockwise or counter-clockwise as the positive direction. It is important to be consistent throughout the circuit!
- Note that not all the equations generated by Kirchhoff's laws are not independent. Only equations for loops that do not include other loops (also called mesh) are independent.

➤ Example - Kirchhoff's Voltage Law (KVL)

Apply Kirchhoff's Current law to find I , V_1 , V_2 , V_3 and the Power Delivered to 24V Source.



• Solution

- (1) Identify meshes (loops that do not include other loops) – there is only one mesh.
- (2) Assign a direction to the current - Clockwise
- (3) Using passive sign convention assign voltage if one is not given. It is important to remember that:
 - (a) Voltage drop across a resistor (passive device) is in the same direction as the current.
 - (b) Voltage drop across a power source (active device) is in the opposite direction as the current.
- (4) Go around the loop and sum up the voltage which is expected to be equal to zero based on Kirchhoff's Voltage Law:

$$-V_g + V_2 + V_3 + V_1 = 0$$

- (5) Use Ohm's Law to substitute V with $R \cdot I$ equivalent. This will reduce the variables from many Voltages to one current:

$$-24 + 3I + 7I + 2I = 0 \rightarrow I = 2 \text{ A}$$

Note: Voltage drop across resistors (+ to -) is always in the direction of arrow regardless of what is being asked for. And if you enter the negative terminal then need to add a negative sign to the voltage value.

Therefore:

$$V_1 = 2I = 4$$

$$V_2 = 3I = 6$$

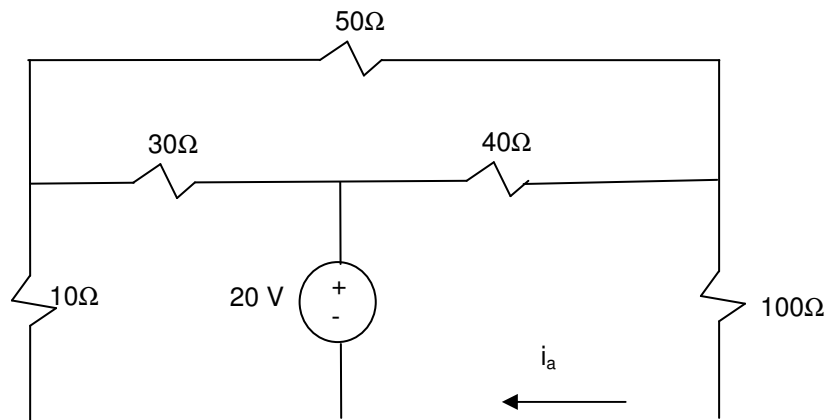
$$V_3 = 7I = 14$$

(6) The only active component is the 24V supply that can deliver power so:

$$\text{Power} = V \cdot I = -(2) \cdot 24\text{V} = -48\text{ W (Power Delivered)}$$

Note: It is multiplied by negative since current is entering into the – side of voltage source

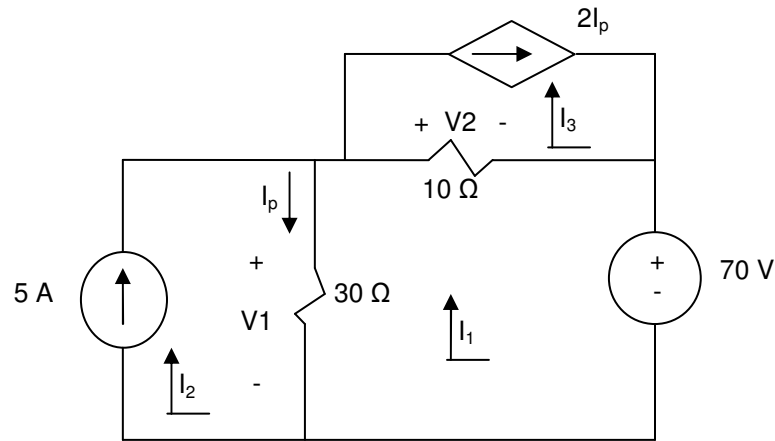
- Example - Kirchhoff's Voltage Law (KVL)
Find i_a in the following circuit by applying KVL:



- Solution(Ans. 112 mA):

- Example - Kirchhoff's Voltage Law (KVL with Dependent source)
When a circuit includes a dependent source there is a need for an additional equation. This is the relationship of the dependent source with the circuit parameter it depends on .

- Find I_p using Krichhoff's Voltage Law (KVL) for the following circuit.



- Identify all the meshes
- Identify all the element voltages
- Add voltages around each loop (be consistent and pick the same direction for all loops)
Note: need to have as many equations as unknowns.
- Insert all the known values in the equations and solve for the unknowns

Solution:

$$\text{Loop 1} \rightarrow 30(I_1 - I_2) + 10(I_1 - I_3) + 70 = 0$$

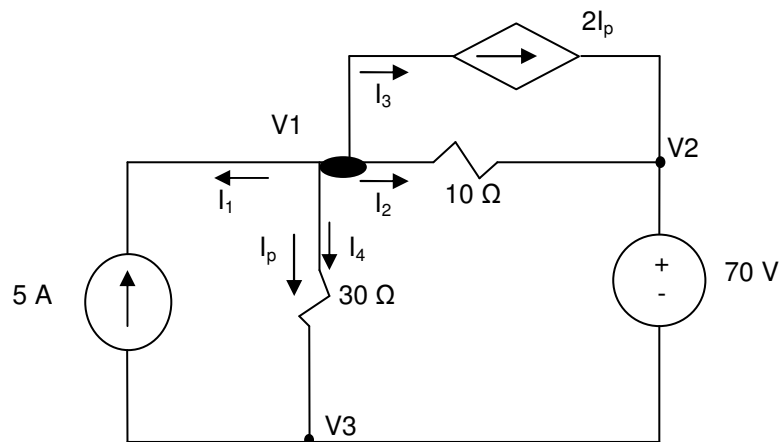
$$\text{Loop 2} \rightarrow I_2 = 5A$$

$$\text{Loop 3} \rightarrow I_3 = 2I_p$$

$$\text{Equation for the Dependent Source} \rightarrow I_p = I_2 - I_1$$

Solving the above 4 simultaneous equation with 4 variables, we get $I_p = 2A \leftarrow$

- Example - Kirchhoff's Voltage Law (KVL)
Find I_p using Kirchhoff's Current Law for the following circuit:



- Find all the essential nodes – three nodes labeled V1, V2 & V3.
Note we can call one a reference and set its value to zero. Let's set V3 as the reference

and set its value to 0 ($V_3 = 0$). As we discussed earlier this assumption will not affect the results but makes the analysis a lot easier.

- (2) Sum the current at each node and set the algebraic sum to zero
As discussed earlier, assume outgoing current is positive. Further we will apply $I = V/R$ to write current in terms of voltage. For example: $I_2 = (v_1 - v_2)/10$.
- (3) Remember that as many independent equations are needed as there are unknowns.
- (4) Solve for the unknowns

Solution:

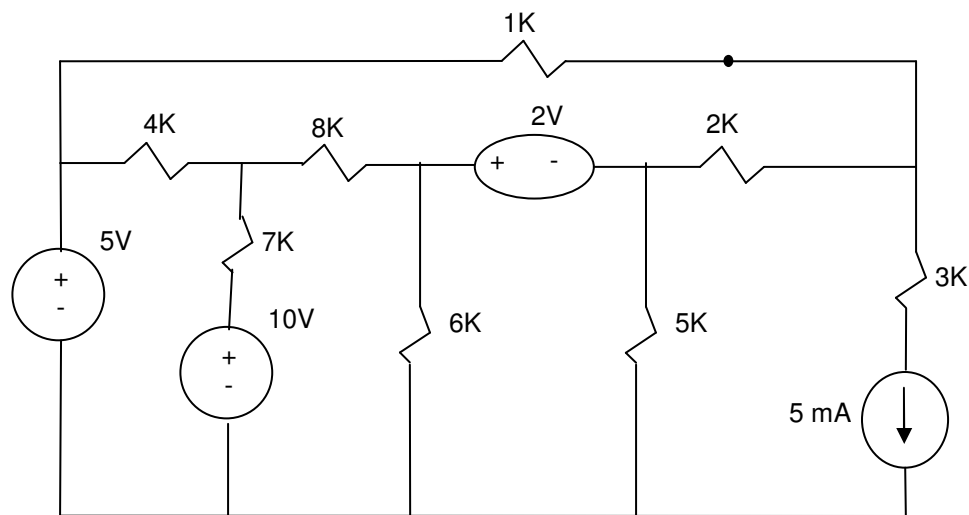
From the circuit $\rightarrow I_1 = -5A$ (Ideal Circuit – Characteristic #1)
 $V_2 = 70V$
 $I_p = V_1/30$

Apply KCL to Node $V_1 \rightarrow I_1 + I_2 + I_3 + I_4 = 0 \rightarrow -5 + (V_1 - 70)/10 + 2I_p + (V_1)/30 = 0$
 Rewrite the above equation with I_p Substitution $\rightarrow -5 + (V_1 - 70)/10 + 2*(V_1)/30 + (V_1)/30 = 0$

Note we have only one equations and one unknown $\rightarrow V_1 = 60$

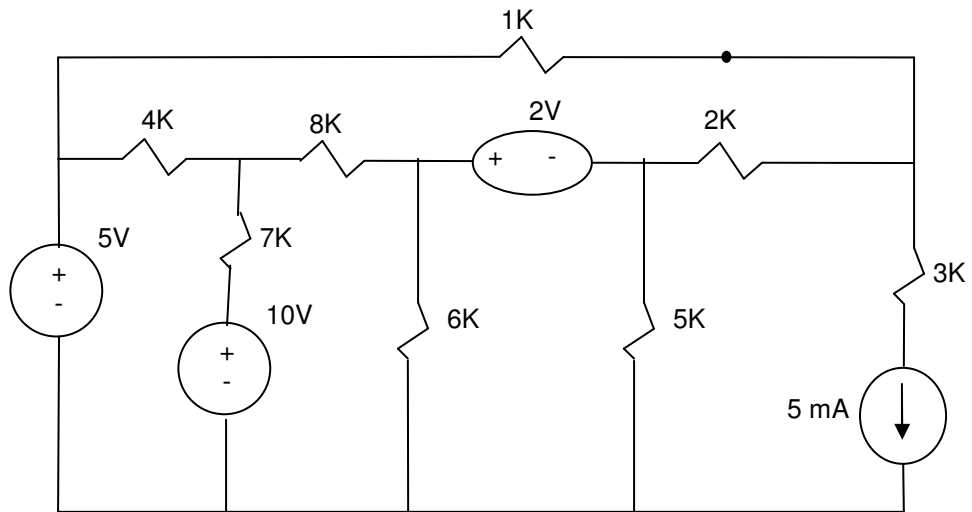
$\rightarrow I_p = V_1/30 = 60/30 = 2$

- Example - Kirchhoff's Voltage Law (KVL with sources in branches)
 Apply KVL to find the value of current in the 2V voltage source in the following circuit:



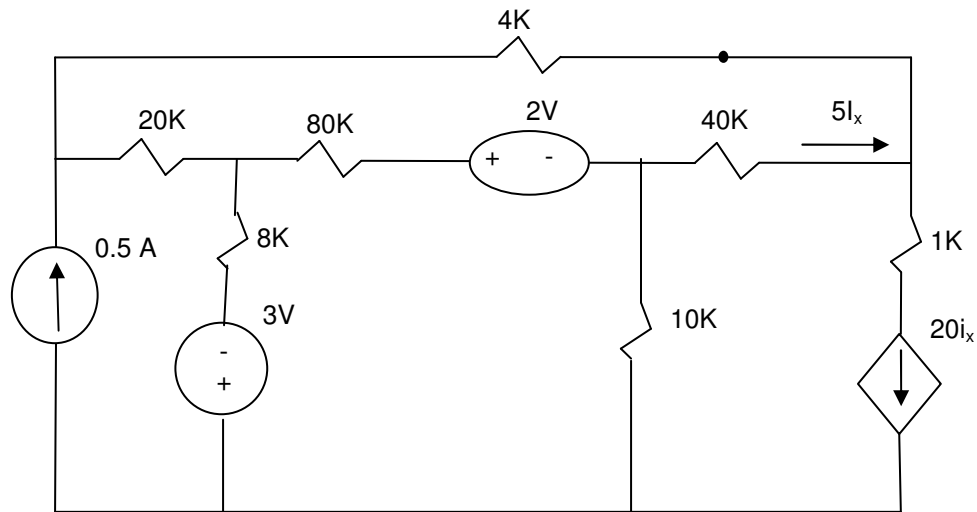
- Solution:

- Example - Kirchhoff's Voltage Law (KVL with sources in branches)
Apply KCL to find the voltage across 5 mA sources in the following circuit:



- Solution:

- Example - Kirchhoff's Voltage Law (KVL with dependent & independent sources)
For the following circuit:



- Apply KVL to find the voltage across 80K resistor.
- Apply KCL to find the voltage across 80K resistor.
- Compare the two approaches.

- Solution:

2.5. Summary

❖ TBC

2.6. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 2 & 3.

2.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 3. Analysis of Resistive Circuits

Key Concepts and Overview

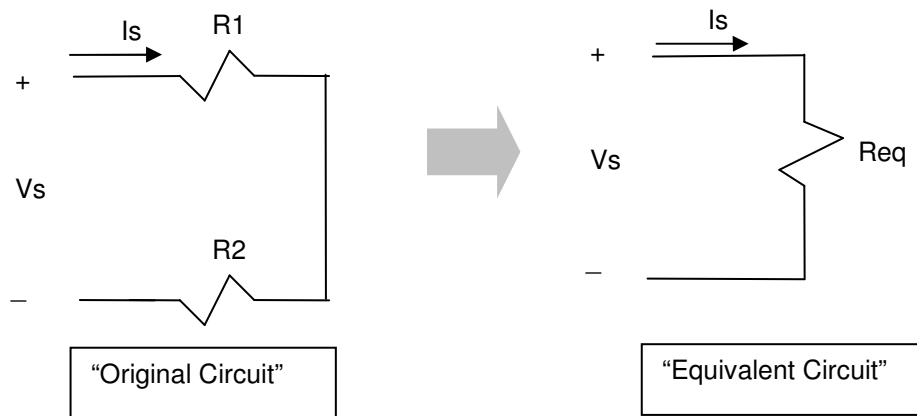
- ❖ Series Resistors Configuration
- ❖ Parallel Resistors Configuration
- ❖ Voltage and Current Divider
- ❖ Measurement Instruments
- ❖ Delta and Wye Configurations
- ❖ Additional Resources

3.1. Series Resistors Configuration

When two resistors connect at a single node and have the same current, they are said to be in series. Combining series resistors simplifies circuit analysis as long as the voltage in individual resistors is not of interest:

We will start with simple case of two resistors in series and expand to the general case of n resistors in series.

- Two resistors in series



- 1) Apply Kirchhoff's voltage law to the original circuit

$$-V_s + I_s \cdot R_1 + I_s \cdot R_2 = 0$$

$$V_s = I_s \cdot (R_1 + R_2)$$

Application of Ohms Law to the equivalent circuit

$$V_s = I_s \cdot R_{eq}$$

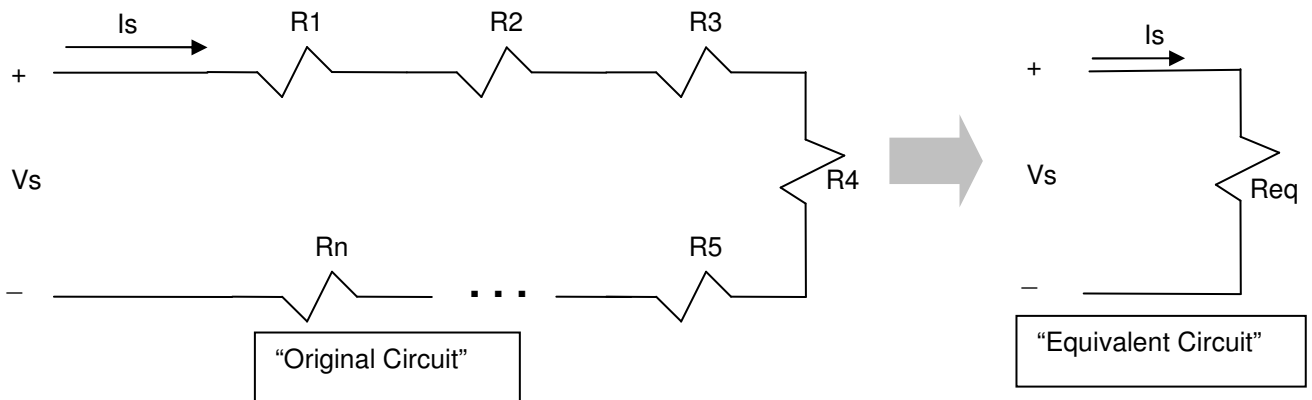
- 2) From above work, we have the following two equations:

$$V_s = I_s \cdot (R_1 + R_2)$$

$$V_s = I_s \cdot R_{eq}$$

We can Conclude that: $R_{eq} = R_1 + R_2$ when R_1 & R_2 are in series.

➤ The General Form



- 1) Apply Kirchhoff's voltage law to the Original circuit

$$-V_s + I_s \cdot R_1 + I_s \cdot R_2 + \dots + I_s \cdot R_n = 0$$

$$V_s = I_s \cdot (R_1 + R_2 + \dots + R_n)$$

Application of Ohms Law to the equivalent circuit

$$V_s = I_s \cdot R_{eq}$$

- 2) From above work, we have the following two equations:

$$V_s = I_s \cdot (R_1 + R_2 + \dots + R_n)$$

$$V_s = I_s \cdot R_{eq}$$

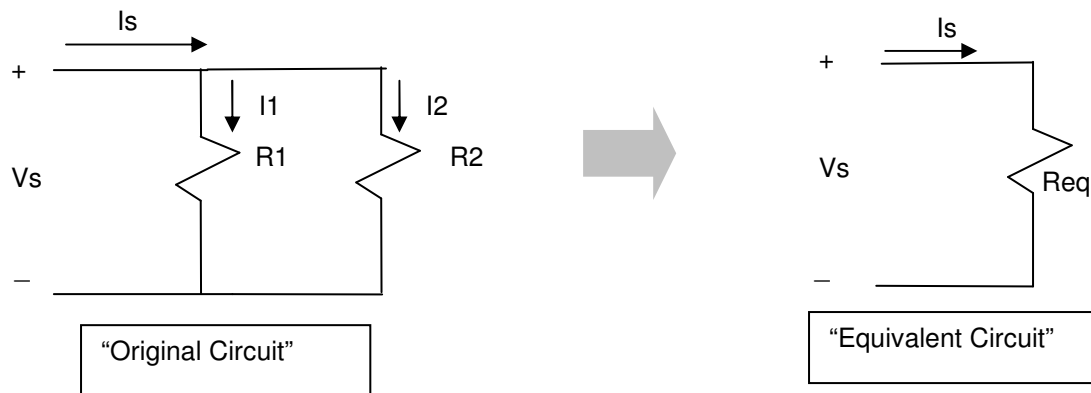
We can conclude that: $R_{eq} = R_1 + R_2 + \dots + R_n = \sum_{i=1}^n R_i$ when R_1, R_2, \dots

3.2. Parallel Resistors Configuration

Two resistors are connected in parallel when the two resistors connect at the same node pair. Combining parallel resistors simplifies circuit analysis as long as the Current in individual resistors is not of interest:

We will start with a simple case of two resistors in parallel and expand to the general case of n resistors in Parallel.

- Two resistors in parallel



- 1) Apply Kirchhoff's current law to the original circuit

$$-I_s + I_1 + I_2 = 0$$

and

Apply Ohms Law to the Equivalent Circuit

$$I_s = V_s / R_{eq}$$

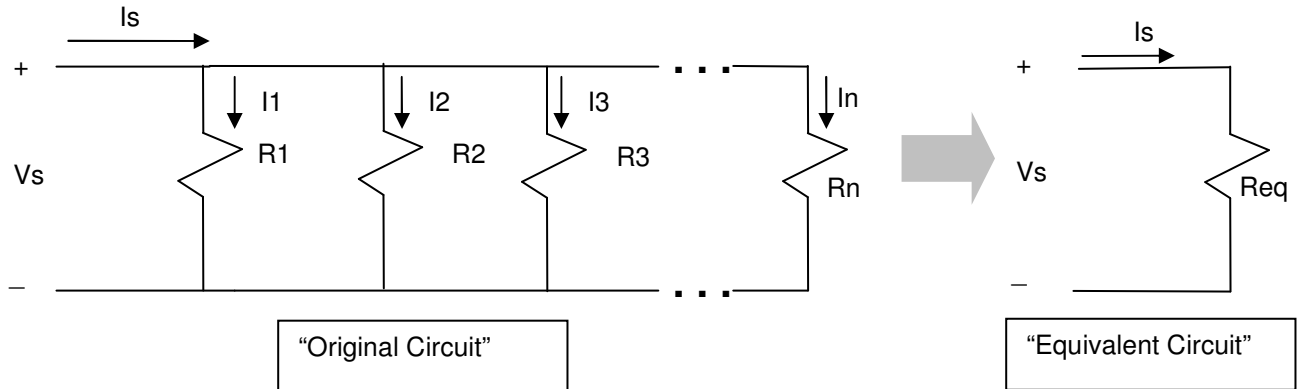
Therefore

$$V_s / R_{eq} = V_s / R_1 + V_s / R_2$$

Divide by V_s

$$1 / R_{eq} = 1 / R_1 + 1 / R_2 \text{ for Parallel Resistors } R_1 \text{ and } R_2.$$

➤ The More General Form



- 1) Apply Kirchhoff's current law to the original circuit

$$-I_s + I_1 + I_2 + I_3 + \dots + I_n = 0$$

and

Apply Ohms Law to the Equivalent Circuit

$$I_s = V_s / R_{eq}$$

Therefore

$$V_s / R_{eq} = V_s / R_1 + V_s / R_2 + V_s / R_3 + V_s / R_n$$

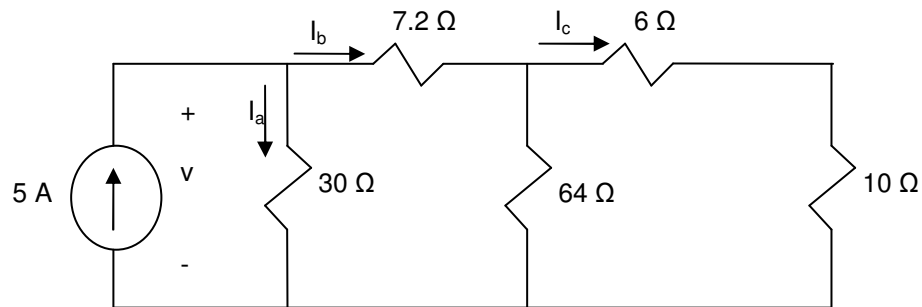
Divide by V_s

$$1 / R_{eq} = 1 / R_1 + 1 / R_2 + 1 / R_3 + \dots + 1 / R_n = \sum_{i=1}^n \frac{1}{R_i}$$

- 2) Sometime it is easier to think of the above relationship in term of Conductance ($G=1/R$)

$$G_{eq} = G_1 + G_2 + \dots + G_n$$

- Example – For the following circuit
- Find the value of voltage v
 - Find the power delivered by the current source

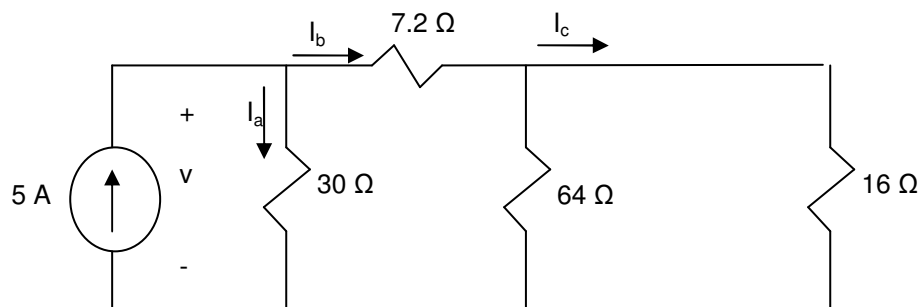


SOLUTION:

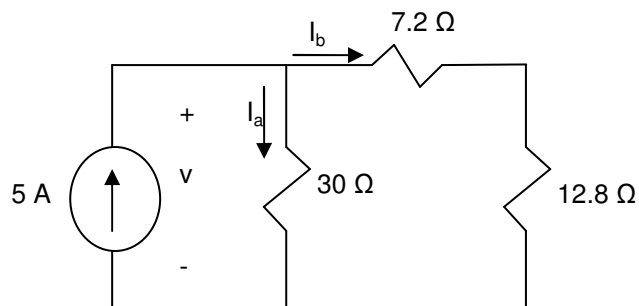
- Find the value of voltage v

Simplify the circuit by finding the Equivalent Resistance (Left to right):

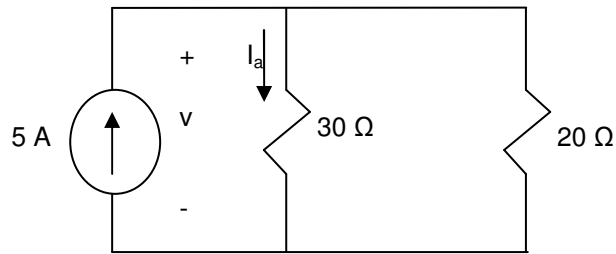
Step 1) 6 Ω is in series with 10 Ω → $R_{eq} = 6 + 10 = 16\ \Omega$ → circuit is redrawn as:



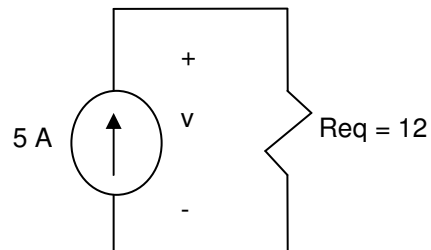
Step 2) 16 Ω is in parallel with 64 Ω with $R_{eq} = (16 \parallel 64) = 12.8\ \Omega$ → circuit is redrawn as:



Step 3) 7.2 Ω is in series with 12.8 Ω with $R_{eq} = (7.2 + 12.8) = 20\ \Omega$ → circuit is redrawn as:



Step 4) $30\ \Omega$ is in parallel with $20\ \Omega$ with $R_{eq} = (20 \parallel 30) = 12\ \Omega \rightarrow$ circuit is redrawn as:



"Note that the 4 steps can be combined as: $R_{eq} = ((10 + 6) \parallel (64)) + 7.2 \parallel (30) = 12\ \Omega$ "

Using the above circuit $\rightarrow v = R_{eq} \cdot I = 12 \cdot 5 = 60\text{ v}$

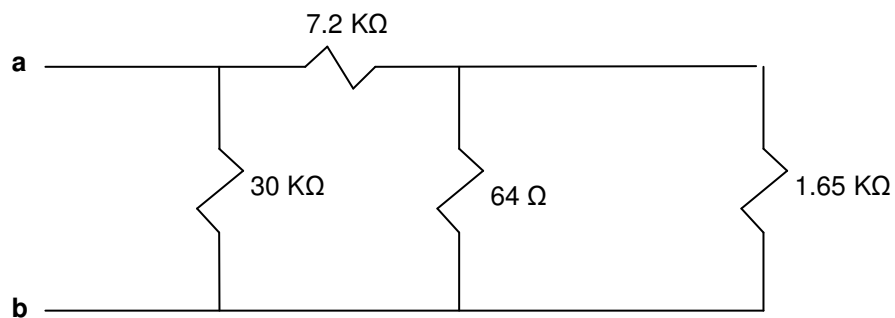
b) Find the power delivered by the current source

$$P = V \cdot I = - (5 \cdot 60) = -300\text{ W}$$

Note: Negative sign was added since current enters the source on "—" side of the voltage.

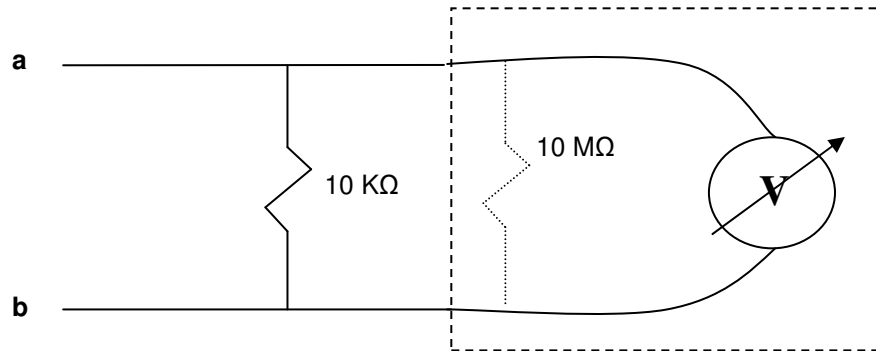
➤ Example – Hints

Determine the resistor equivalent at point a and b for the following circuit:



▪ Solution

- Example – Hints
Determine the percent measurement error if a voltmeter with $10\text{ M}\Omega$ internal resistance is used to measure the $10\text{ K}\Omega$ resistor:



- Solution

3.3. Voltage and Current Dividers

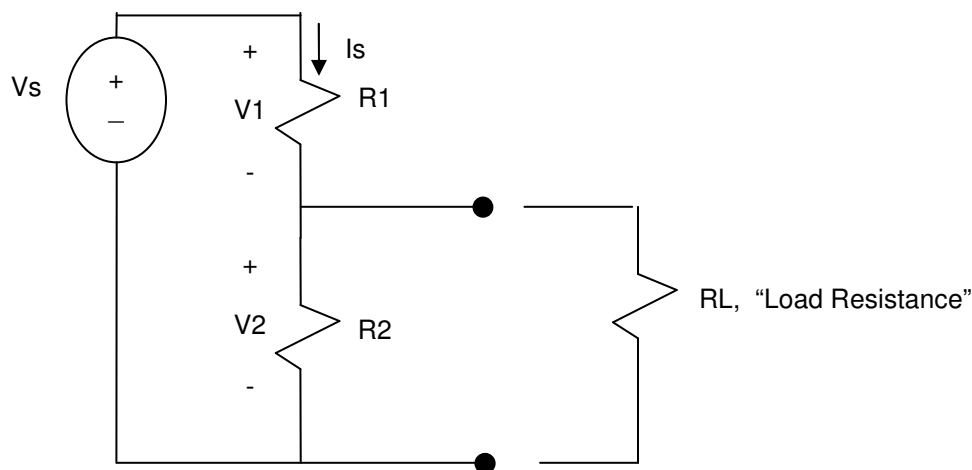
Voltage and Current Dividers are circuits for developing multiple lower voltage and current levels from a fixed source by utilizing appropriately sized resistors. Although similar in concept, there are enough differences between voltage and current dividers that it is beneficial to discuss each individually.

❖ Voltage-divider Circuit

A voltage divider is a combination of resistors that produces two or more voltages from a single power supply. The concepts developed in parallel and series resistor simplification apply here also.

➤ Producing the desired voltage using two resistors

The two cases considered here include one without load ($R_L = \infty$) and the other with load ($R_L < \infty$). Load represents the device being powered by the power source.



➤ Without Load

Apply Kirchhoff's Voltage Law $\rightarrow -V_s + I_s * R_1 + I_s * R_2 = 0 \rightarrow I_s = V_s / (R_1 + R_2)$

Therefore:

1) $V_1 = I_s * R_1 = V_s * R_1 / (R_1 + R_2)$

2) $V_2 = I_s * R_2 = V_s * R_2 / (R_1 + R_2)$

➤ With the Load

R_2 has to be replaced by Resistor equivalence of parallel combination of R_2 and R_L

$$R_{eq} = 1 / (1/R_2 + 1/R_L) = (R_L * R_2) / (R_L + R_2)$$

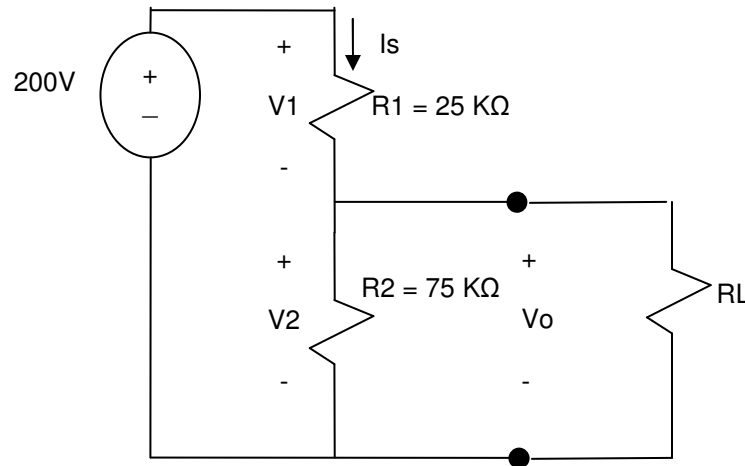
replacing R_2 with R_{eq} in V_2 equation:

$$V_2 = V_s * \{(R_L * R_2) / (R_L + R_2)\} / \{R_1 + (R_L * R_2) / (R_L + R_2)\}$$

➤ The concept of a voltage divider can be extended to n resistors in series creating n different voltages. Using:

$$V_i = V_s * R_i / R_{eq} \text{ for the voltage across the } i^{\text{th}} \text{ resistor.}$$

- Example – Voltage Divider
For the following circuit



- Find the no-load value of V_o in the circuit above. ($R_L = \infty$).
- Find V_o when R_L is 150 kΩ.
- How much power is dissipated in the 25 kΩ if the load is short circuited ($R_L = 0$)?
- What Value of R_1 delivers the maximum power to the 75KΩ resistor?

Solution:

- Find the no-load value of V_o in the circuit above. ($R_L = \infty$).

$$V_o = 200 \cdot 75 / (25 + 75) = 150 \text{ V}$$
- Find V_o when R_L is 150 kΩ

$$R(75 \parallel 150) = 1 / ((1/75) + (1/150)) = 50 \text{ k}\Omega$$

$$V_o = 200 \cdot 50 / (25 + 50) = 133.33 \text{ V}$$
- How much power is dissipated in the 25 kΩ if the load is short circuited ($R_L = 0$)?

$$R(75 \parallel 0) = 0 \text{ }\Omega$$

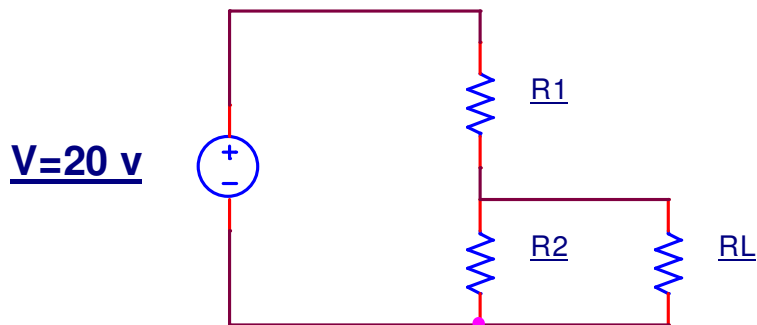
$$V_1 = 200 \cdot 25 / (25 + 0) = 200 \text{ V} \text{ (The whole 200 V is seen by the 25 K}\Omega\text{)}$$

$$P(25\text{K}\Omega) = V_1^2 / R = (200)^2 / 25,000 = 1.6 \text{ W}$$
- What Value of R_1 delivers the maximum power to the 75KΩ resistor?

$$R_1 = 0 \rightarrow \text{deliver max power to 75K}\Omega$$

$$P(75\text{K}\Omega) = V_o^2 / R = (200)^2 / 75,000 = 0.53 \text{ W}$$

- Example – Voltage Divider
For the following Circuit:



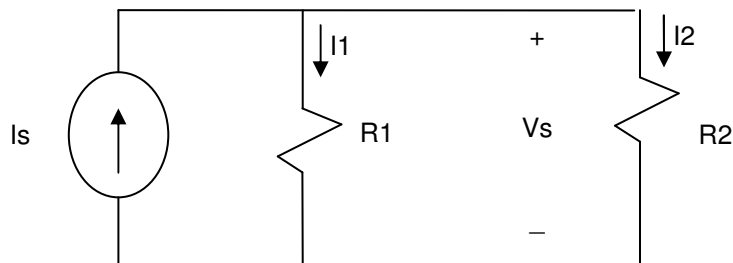
- Find value of R_1 and R_2 such that voltage across R_2 is 5v ($R_L = \infty$).
- What are the new values of R_1 and R_2 if you only have access to 1-20 M Ω resistors?
- What are the new values of R_1 and R_2 if you only have access to 1-20 Ω resistors?
- What is the difference in power required from the voltage source between parts b and c?
- How does the answers to part b and c changes if $R_L = 1$ k Ω ?
- What are the optimal resistors (R_1 and R_2) if $R_L = 1$ k Ω , use minimum power and keep voltage across R_L equal to 5V $\pm 10\%$.

Solution:

❖ Current-divider Circuit

A current divider is a combination of resistors that produces two or more current levels from a single power source. The concepts are similar to the ones used in designing/analyzing voltage dividers.

- Producing the desired current using two resistors

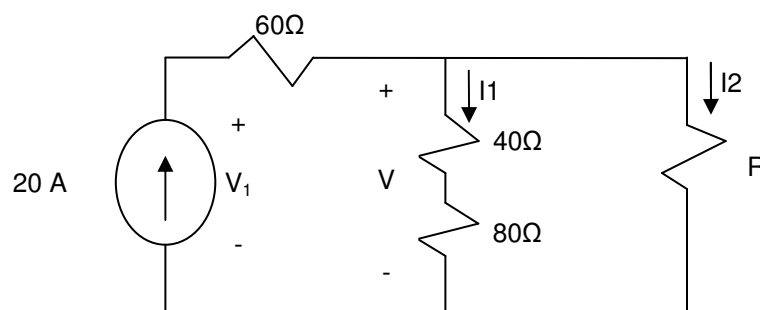


- 1) Apply Kirchhoff's current Law and combining resistors →
 $V_s = I_s \cdot R_{eq} = I_s \cdot (R_1 \cdot R_2 / (R_1 + R_2))$
 which results in two current levels inversely proportional to resistor sizes:
 $I_1 = V_s / R_1 = (I_s \cdot R_2) / (R_1 + R_2)$
 $I_2 = V_s / R_2 = (I_s \cdot R_1) / (R_1 + R_2)$

- The concept of a current divider can be extended to n resistors in parallel creating n different currents.

$$I_i = I_s \cdot R_{eq} / R_i \text{ for the current through the } i^{\text{th}} \text{ resistor}$$

- Example – For the following circuit
 - a) Find the value of R that will cause 4 A of current to flow through the 80 Ω resistor in the above circuit.
 - b) How much Power will R from part a need to dissipate?
 - c) How much power will the current source generate for the value of R from part a?



- 1) Solution:
 - a) Find the value of R that will cause 4 A of current to flow through the 80 Ω resistor in the above circuit.

$$(1) V = I_1 \cdot (40 + 80) = 4 \cdot (120) = 480 \text{ V}$$

$$(2) \text{ Apply KCL } \rightarrow -20 + 4 + I_2 = 0 \rightarrow I_2 = 16 \text{ A}$$

$$(3) \quad V = I_2 R \rightarrow 480 = 16 * R \rightarrow R = 30 \text{ Ohms}$$

- b) How much power will R calculated in part (a) need to dissipate?

$$P = I_2^2 R = 16^2 * 30 = 7680 \text{ W}$$

- c) How much power will the current source generate for the value of R calculated in part a?

$$V_1 = V + 60 * 20 = 480 + 1,200 = -1,680 \text{ V}$$

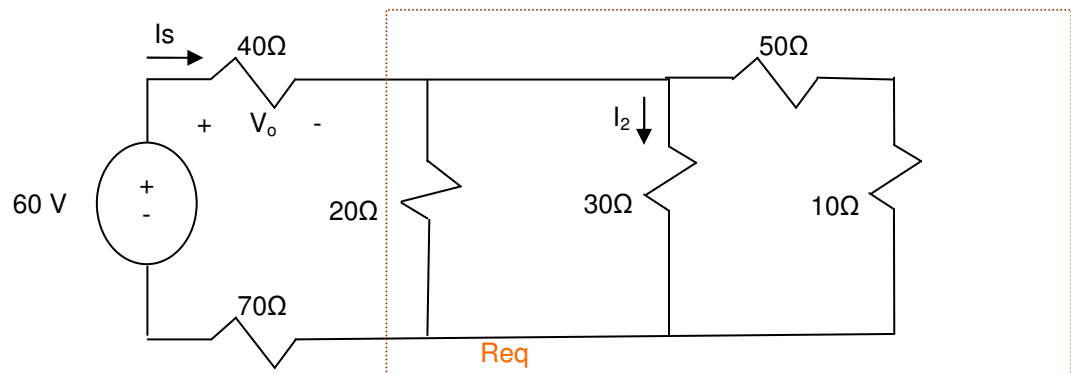
$$P_{(20A \text{ source})} = -V * I = -1,680 * 20 = -33,600 \text{ W (Generated Power)}$$

Note: Negative since current is entering the negative side of the voltage.

- Example – For the following circuit

- a) Find Voltage V_o

- b) Find current through the 30Ω resistor.



- 1) Solution:

- a) Find the Voltage V_o

$$Req1 = (((10+50) \parallel 30) \parallel 20) = 1/(1/60 + 1/30 + 1/20) = 10 \Omega$$

$$\text{Apply voltage divider rules } V_i = V_s * R_i / R_{eq} \rightarrow V_o = (60)(40/(40+10 + 70)) = 20V$$

- b) Find current through 30Ω .

$$I_s = V_o/40 = 20/40 = 0.5A$$

Apply Current divider rule $I_i = I_s * R_{eq} / R_i$ note I_s to be shared among 20, 30 and (50+10)

ohm resistors. So $R_i=30$ and $Req2 = (20 \parallel 30 \parallel 60) = 1/(1/20 + 1/30 + 1/60) = 10$

$$I_2 = I_s * Req2/30 = 0.5 * 10 / 30 = 0.168 \text{ A}$$

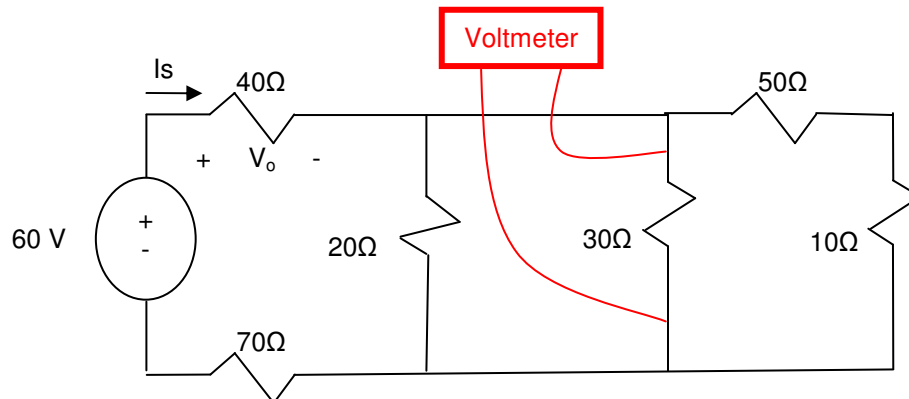
3.4. Measurement Instruments

Three important measurement instruments in electrical engineering are Voltmeters, Ammeters and Ohmmeters. In the past these instruments were designed using a device called d'Arsonval meter movement. Modern instruments use analog-to-digital conversion to implement all three functions and are called Digital MultiMeters (DMMs).

❖ Voltmeter

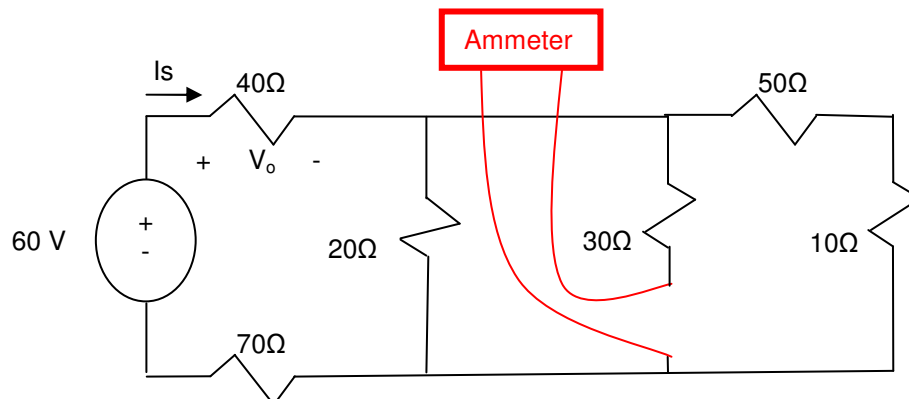
Voltmeters are designed to measure voltage across a portion of the circuit and present a very high resistance ($R \approx \infty$). This characteristic allows the voltmeter to be placed across a pair of nodes without impacting the behavior of the circuit since $R_{eq} = R1 \parallel \infty = R1$.

So to measure voltage across the $30\ \Omega$ resistor, place the voltmeter in parallel as shown below:



❖ Ammeter

Ammeters are used to measure current through a branch of a circuit and are designed to present a very low resistance ($R \approx 0$). This characteristic allows the Ammeter to be placed between a pair of nodes without impacting the behavior of the circuit since $R_{eq} = R1 + 0 = R1$.



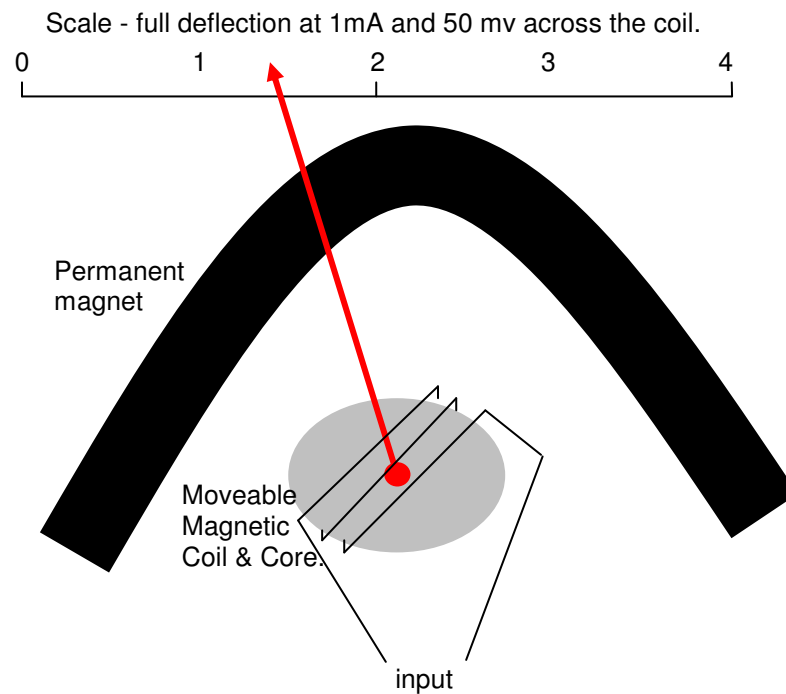
❖ Digital Multi-Meter (DMM)

Digital Multi-Meters, as the name implies are digital. DMMs sample a signal periodically and convert the reading from analog to digital in order to display it. DMMs typically include Ammeter, Ohmmeter and Voltmeter functionality. By selecting the appropriate function, the connection to the circuit will be of the same as describe in the earlier description of Ammeter and Voltmeter usage.

❖ Pre-digital application -- Analog Meters (also called Volt Ohm Meter, VOM)

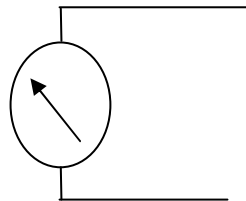
These analog instruments relied on d'Arsonval meter movement for visualizing of the data. The mechanism relied on flow of current to magnetize a dial that would move to show the measure of the current and voltage.

Below is a functional diagram of an instrument using d'Arsonval meter movement.

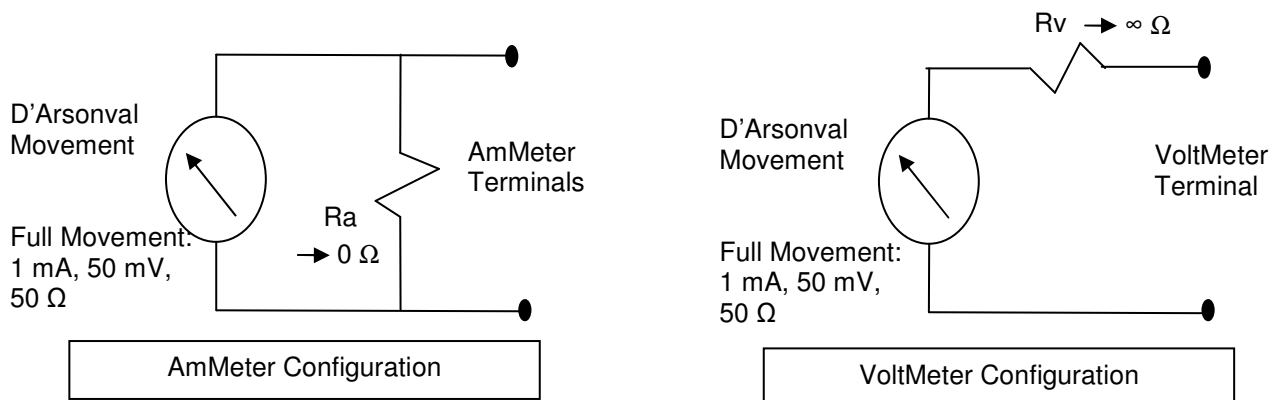


The d'Arsonval meter movement is typically designed to require input of 50 mV and 1mA in order for the pointer to move to the full-scale position. The d'Arsonval meter movement is typically represented as:

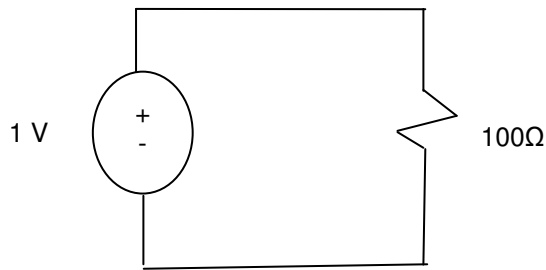
D'Arsonval Movement
Full Movement: 1 mA, 50 mV, 50 Ω



A d'Arsonval meter movement along with a resistor can be used to design analog Ammeter or Voltmeter as shown below:



❖ Example - For the following circuit



a) Find the current in the circuit.

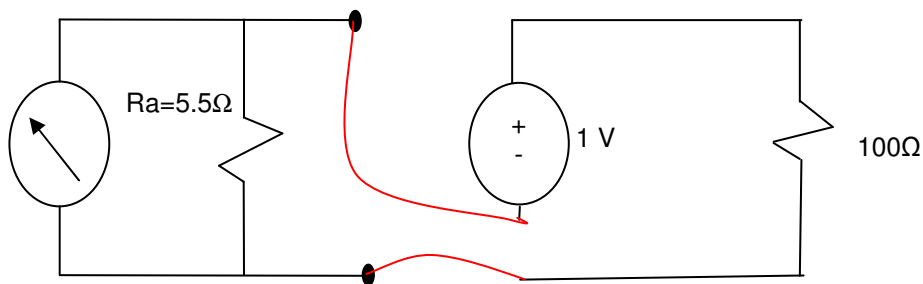
b) What would an Ammeter using the d'Arsonval meter movement ($R_a = 5.5 \text{ ohms}$) read?

➤ Solution:

a) Find the current in the circuit.

$$i = V/R = 1/100 = 10 \text{ mA}$$

b) what would an Ammeter using the d'Arsonval meter movement ($R_a = 5.5 \text{ ohms}$) read?



$$I = (1) / (100 + 5.5) = 9.5 \text{ mA}$$

❖ Another pre-digital Application - The Wheatstone Bridge

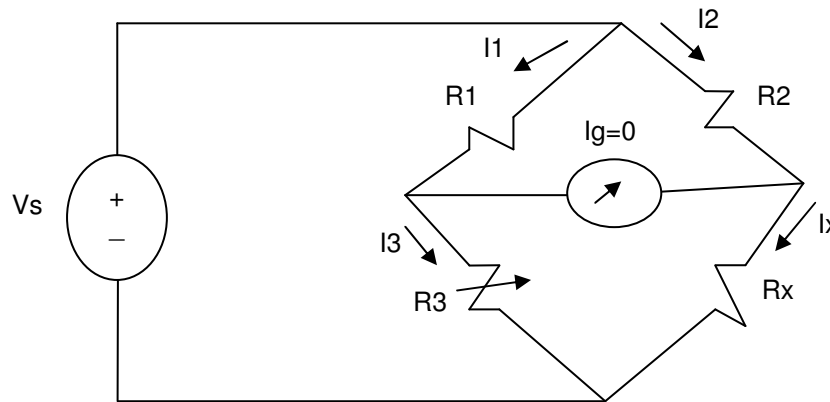
The Wheatstone Bridge was used to measure resistance using the following components and procedure:

1) Components

- (1) Resistor to be measured (R_x)
- (2) a set of 2 fixed resistors R_2 and R_1 with R_2/R_1 ratio from 0.001 to 1000 driven by the size of resistor being measured.
- (3) Variable resistor R_3 from 1 to 11,000 Ω with a physical dial that shows the value of the resistor.
- (4) Ammeter using D'Arsonval movement
- (5) Voltage Source

2) Procedure

- (1) Adjust variable resistor until no current flow through Ammeter ($I_g=0$)
- (2) Calculate value of $R_x = (R_2/R_1) \cdot R_3$



- Proof that $R_x = (R_2/R_1) \cdot R_3$
 - 1) By applying Kirchhoff's law and the fact that $I_1 = I_3$ & $I_2 = I_x$ since R_3 is adjusted so that $I_g = 0$
 - 2) Top Triangle $\rightarrow I_1 \cdot R_1 = I_2 \cdot R_2$ (Since $I_g = 0$)
 - 3) Bottom Triangle $\rightarrow I_3 \cdot R_3 = I_x \cdot R_x$ (Since $I_g = 0$)
 - 4) Divide both side and remember that and $I_1 = I_3$ & $I_2 = I_x$ since R_3 is adjusted so that $I_g = 0$
 $R_1/R_3 = R_2/R_x \rightarrow R_x = (R_2/R_1) \cdot R_3$
- Accuracy

The accuracy of the measurement by the Wheatstone Bridge depends on a number of factors as outlined below:

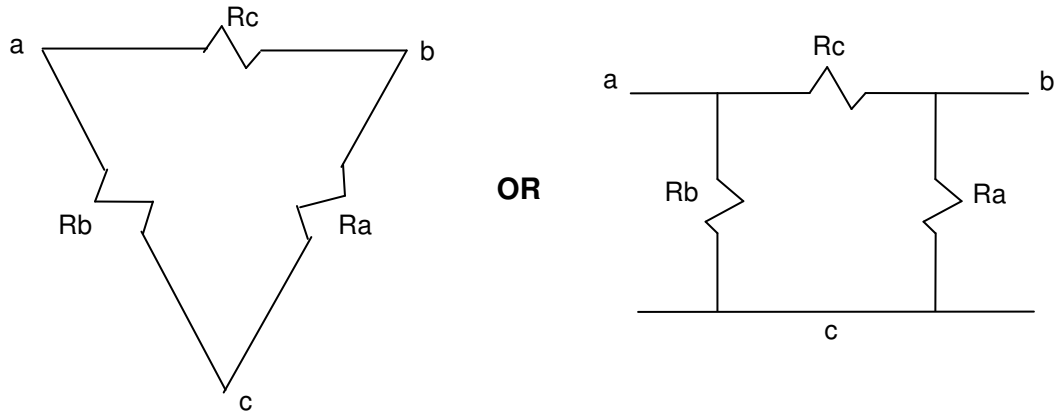
 - 1) Resistors with values below 1Ω are too small to be measured with standard Wheatstone due to:
 - (1) Thermoelectric voltage generated at the junction of dis-similar metal
 - (2) Thermal heating effects (resistance increases as temperature goes up due to power consumption $I^2 \cdot R$)
 - 2) Higher value resistors' measurements are affected by the leakage current. Leakage current is any current through I_g , even if it is small. This error will be amplified proportionally to the size of the large resistors being measured.

3.5. Delta and WYE Configurations

Delta and Wye are two other resistor network configurations that are neither the series nor the parallel configurations that were discussed earlier. These two configurations, Delta and Wye are shown in the following diagrams:

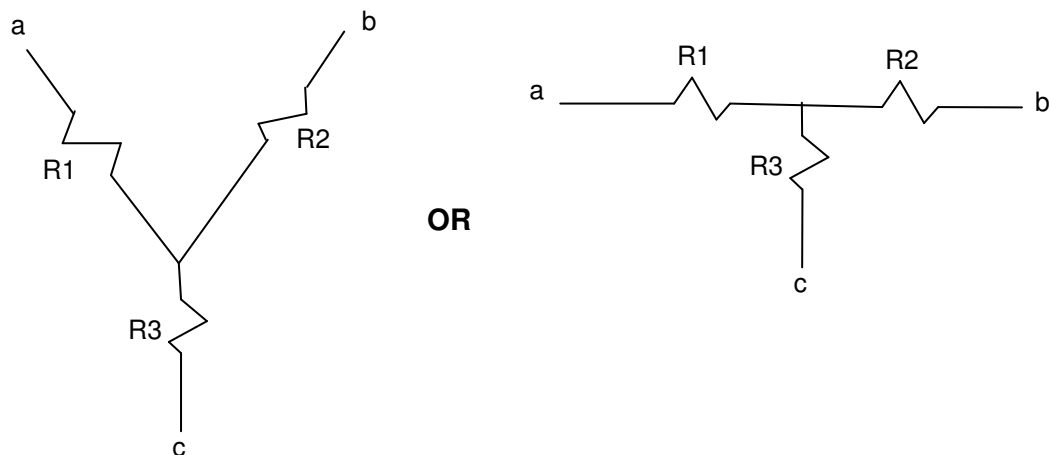
➤ Delta (Δ) or Pi (π) Configuration

The following diagrams show two ways to represent a Delta configuration:



➤ Wye (Y) and Tee (T) Configuration

The following diagrams show two ways to represent a Y configuration:



From time-to-time, conversion between Delta and Wye leads to circuit simplification. Three equations are needed in order to convert from one to its equivalent circuit in the other forms,. These equations are derived from setting the equivalent circuit for each node pairs in one configuration equal to the corresponding pair in the other configuration as shown below for node pairs ab, bc and ac:

$$R_{ab} = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} = R_1 + R_2$$

$$R_{bc} = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} = R_2 + R_3$$

$$R_{ac} = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c} = R_1 + R_3$$

By applying algebra to the above equations, each resistance can be found in-term of other resistances such as:

➤ Delta-to-Wye Conversion

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \quad R_2 = \frac{R_c R_a}{R_a + R_b + R_c} \quad R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

➤ Wye-to-Delta Conversion

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_1} \quad R_b = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_2} \quad R_c = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_3}$$

3.6. Summary

❖ TBC

3.7. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 3.

3.8. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 4. Circuit Analysis/Modeling Techniques

Key Concepts and Overview

- ❖ Introduction
- ❖ Node-voltage Method (based on KCL)
- ❖ Mesh-current Method (based on KVL)
- ❖ Node-Voltage and Mesh-Current Comparison
- ❖ Source Transformation
- ❖ Thévenin & Norton Equivalents
- ❖ Delivering Maximum Power
- ❖ Superposition Principle
- ❖ Sensitivity Analysis
- ❖ Additional Resources

4.1. Introduction

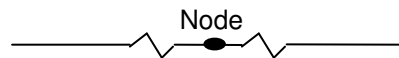
This chapter introduces more generalized analysis techniques such as Mesh-Current, Node-Voltage, Source Transformation, Thevenin and Norton Equivalent and Super-Position. These techniques can be applied to circuits with large numbers of elements. Many of these techniques are used in Computer Aided Design and Analysis tools for circuit analysis. All the methods introduced here rely on the three fundamental laws of Electrical Circuits which were introduced in earlier chapters:

- 1) Ohm's Law
- 2) Kirchhoff's Current Law (KCL)
- 3) Kirchhoff's Voltage Law (KVL)

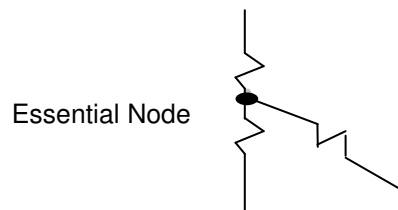
❖ Basic Circuit Analysis Terms

It is time to introduce the formal definition of key terms that will be used throughout Electrical Circuit Analysis:

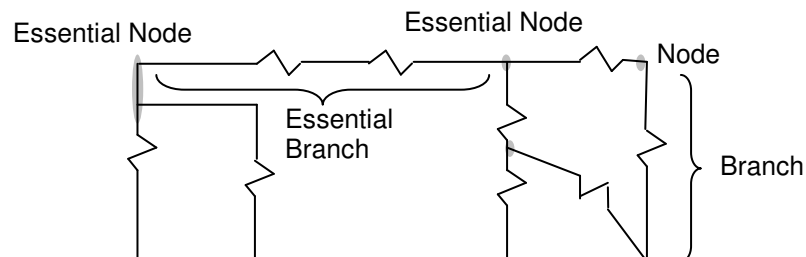
- Node: A point where two or more circuit elements join.



- Essential Node: A node where three or more circuit elements join. In a circuit analysis text, it is common to refer to essential nodes as simply nodes. This inaccuracy is accepted since we are only interested in essential nodes in circuit analysis.



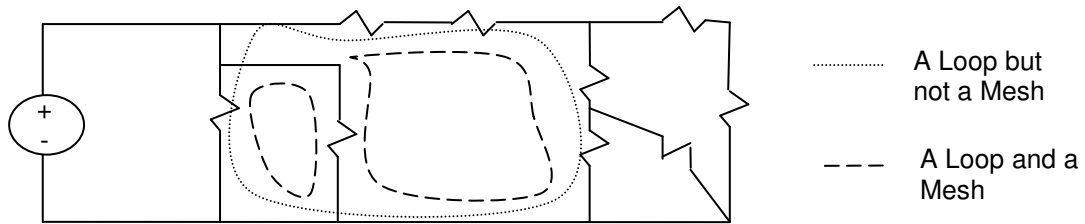
- Path: A trace through adjoining basic elements where no element is traced more than once.
- Branch: A path that connects two nodes.
- Essential Branch: A path which connects two essential nodes without passing through another essential node.



In course of circuit analysis, it is common to refer to essential branches as simply branches. This inaccuracy is accepted since generally we are only interested in essential branches in circuit analysis.

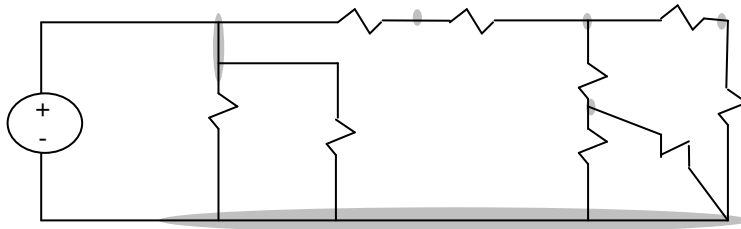
- Loop: A path where the end of the path is at the starting node.

- Mesh: A loop that does not enclose any other loop.



- Planar Circuit: A circuit that can be drawn on a plane with no crossing branches. In order to apply the circuit analysis techniques introduced in this text, the circuit must be planar and drawn such that no branch is crossing another branch of the circuit.

- ❖ Example – Application of Definition of Term
Apply the above definitions to the following Circuit:



- This is a Planar Circuit since none of the branches cross each other.
- Node = 6
- Essential Node = 4
- Path > 10
- Branch = 10
- Essential Branch = 8
- Loop > 10
- Mesh = 5

4.2. Node-Voltage Method (Based on KCL)

Node-Voltage Method is a formalized and systematic approach to the application of KCL to circuit analysis. This method is applicable to highly complex circuits due to the fact that is a systematic approach. The following steps describe Node-Voltage Method:

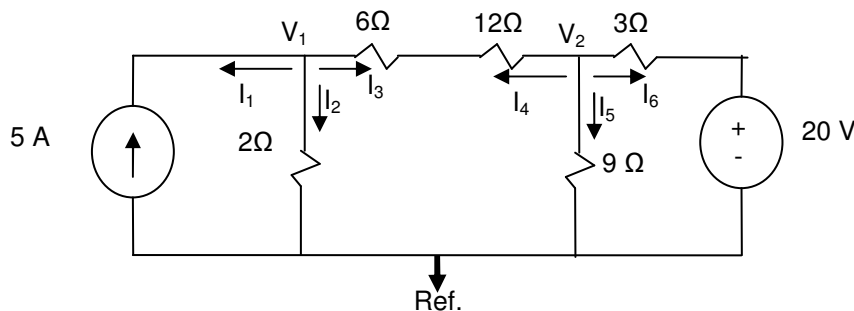
- 1) Redraw the circuit as a planar circuit
- 2) Identify essential nodes (n)
- 3) Select one essential node as reference and mark it with "↓ Ref"
The voltage at the reference node is assumed to be zero. Any of the essential nodes may be selected as reference since all the voltages being calculated are relative. The reference node is typically the node with highest number of branches which results in the simplest analysis.
- 4) Number of independent equations resulting from application of KCL to all nodes except reference node is (n-1).
- 5) Define Voltage at every node (V_1, V_2, \dots, V_n)
- 6) Using KCL, write the sum of current in-terms of voltage for every node except reference.

Hints:

- a. Assume current going out of a node is positive
- b. If a branch contains a resistor, the current through the branch is the voltage difference across the resistor divided by the resistance.
- c. If the branch contains a current source, the current of the current source is also the branch current.

❖ Example - Node-Voltage Method Application

For the circuit shown below , find the value of voltage across the 5A current source.



Solution:

Circuit is drawn as planar, essential node voltages are marked with V_1, V_2 and the reference node ($V_{Ref}=0$)

Write the KCL for each essential nodes (V_1, V_2) except reference node:

$$\text{Node 1} \rightarrow I_1 + I_2 + I_3 = 0 \rightarrow -5 + V_1/2 + (V_1 - V_2)/(6 + 12) = 0$$

$$\text{Node 2} \rightarrow I_4 + I_5 + I_6 = 0 \rightarrow (V_2 - V_1)/(6 + 12) + (V_2)/9 + (V_2 - 20)/3 = 0$$

Simplify and rewrite the equations:

$$-90 + 10V_1 - V_2 = 0$$

$$-120 - V_1 + 9V_2 = 0$$

Solve the above two simultaneous equations $\rightarrow V_1 = 10.45 \text{ V}$ & $V_2 = 14.5 \text{ V}$

Therefore Voltage across 5A current source $\rightarrow V_1 = 10.45 \text{ V}$

❖ Node-Voltage Method Special Cases

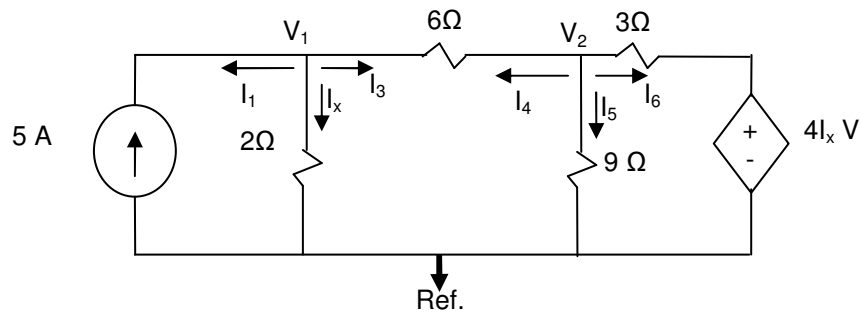
This section lists special cases that are encountered when using the Node-Voltage Method for analyzing the circuits.

➤ Special Case 1 - Circuits with Dependent Source

When the circuit being analyzed includes a dependent source, the analysis includes an additional variable which is the current or voltage controlling the dependent source.

The analysis of this type of circuit is similar to the Node-Voltage method described earlier with the added requirement of writing an equation to relate the branch currents to the dependent source variable.

For example, the following circuit contains a dependent voltage source that depends on the variable current I_x :



In order to find the node voltages and I_x using Node-Voltage Analysis, first apply all the Node-Voltage Method steps and write the resulting KCL equations:

$$\begin{aligned} \text{Node 1} &\rightarrow I_1 + I_2 + I_3 = 0 \rightarrow -5 + V_1/2 + (V_1 - V_2)/6 = 0 \\ \text{Node 2} &\rightarrow I_4 + I_5 + I_6 = 0 \rightarrow (V_2 - V_1)/6 + (V_2)/9 + (V_2 - 4I_x)/3 = 0 \end{aligned}$$

Simplify and rewrite the equations:

$$\begin{aligned} -90 + 4V_1 - V_2 &= 0 \\ -24I_x - V_1 + 11V_2 &= 0 \end{aligned}$$

The above two equations have three unknowns (V_1 , V_2 and the dependent source variable, I_x).

The third equation is derived from the circuit $\rightarrow I_x = V_1 / 2$

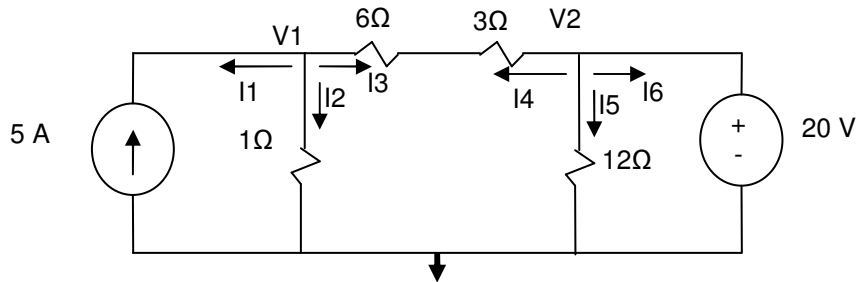
Now, there are three equations and three unknowns so it can be solved.

$$\begin{aligned} 4V_1 - V_2 &= 90 \\ -24I_x - V_1 + 11V_2 &= 0 \\ 2I_x - V_1 &= 0 \end{aligned}$$

Solving the system of equations $\rightarrow I_x = 16.0 \text{ A}$, $V_1 = 31.9 \text{ V}$, $V_2 = 37.7 \text{ V}$

- **Special Case 2 – Circuit with a Branch Containing Only a Voltage Source**
 When a circuit contains a branch with only a voltage source, Ohms Law ($I=V/R$) cannot be applied to find branch current. This means that the KCL equation for the node with such a branch cannot be written in term of voltage. The good news is that you do not need to write the KCL equation since the node voltage is already known and is equal to the value of the voltage source.

For Example, in the analysis of the following circuit:



There is no need to write a KCL equation for Node 2 since $V_2 = 20$. This means the system of equations is reduced to:

$$\begin{aligned}\text{Node 1} &\rightarrow -5 + V_1 + (V_1 - v_2)/(3 + 6) = 0 \rightarrow 45 + 10V_1 - V_2 = 0 \\ \text{Node 2} &\rightarrow V_2 = 20 \text{ V}\end{aligned}$$

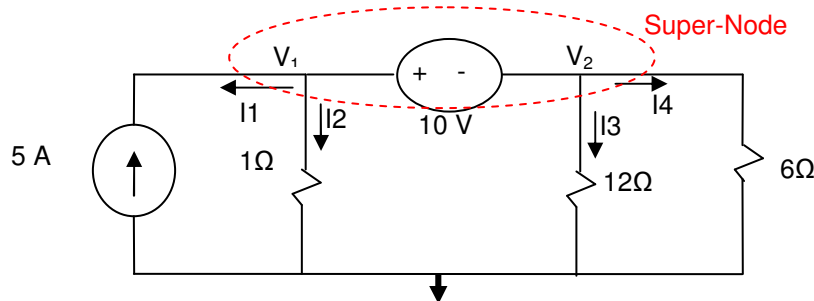
Which is a much simpler system of equation to solve. $\rightarrow V_2 = 20 \text{ V}$, $V_1 = 6.5 \text{ V}$

- **Special Case 3 – Super-Node**
 When a circuit contains a voltage source between two essential nodes, it is possible to combine the two essential nodes as one Super-Node. Only one KCL is needed for the Super-Node since the relation between the two node voltages is simply the value of the voltage source.

The Super-Node concept may be applied for dependent and independent voltage sources between two essential nodes: The Node-Voltage steps apply to circuits with the additional requirements to:

- (1) Draw a circle around the two nodes and call it a super node; there is no need to place voltage label on it.
- (2) Write the KCL equations as if circle area is one node (Super Node KCL equation)
- (3) Write another equation that relates the two node voltages in the Super node

For example in the following circuit, nodes V_1 and V_2 can be combined into a Super-Node as shown here:



$$\begin{aligned}\text{Super-Node KCL} &\rightarrow I_1 + I_2 + I_3 + I_4 = 0 \rightarrow -5 + V_1 + V_2/12 + V_2/6 = 0 \\ \text{The difference between the nodes is the voltage source voltage} &\rightarrow V_1 - V_2 = 10\end{aligned}$$

Simplify the above two equations:

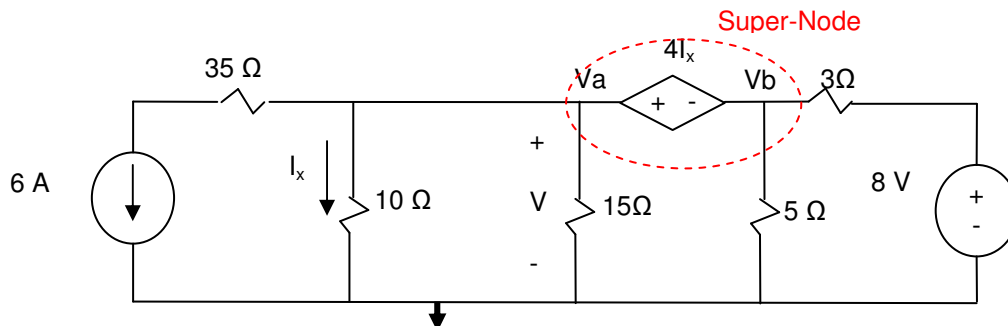
$$12V_1 + 3V_2 = 60$$

$$V_1 - V_2 = 10$$

Solve the two equations $\rightarrow V_1 = 6 \text{ V}; V_2 = -4 \text{ V};$

❖ Examples – Application of General Node-Voltage Method

➤ Example 1 - Use the Node-Voltage Method to find V in the circuit shown.



Solution:

$$\text{Super-Node KCL} \rightarrow 6 + V_a/10 + V_a/15 + V_b/5 + (V_b - 8)/3 = 0$$

$$\text{Super-Node internal Nodes' voltage relationship} \rightarrow V_a - V_b = 4I_x$$

$$\text{Control source Variable Equation} \rightarrow I_x = V_a/10$$

Simplify the 3 equations:

$$5V_a + 16V_b = -100$$

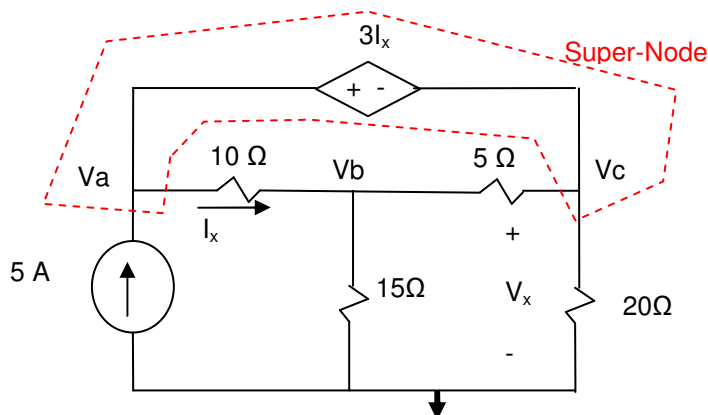
$$-4I_x + V_a - V_b = 0$$

$$10I_x - V_a = 0$$

Solve the system of 3 equations and 3 unknowns:

$$I_x = 2.3 \text{ A}; V_a = 2.3 \text{ V}; V_b = -7 \text{ V} \rightarrow V = V_b = -7 \text{ V}$$

➤ Example 2 - Use the Node-Voltage Method to find V_x in the following circuit.



Solution:

$$\text{Super-Node KCL} \rightarrow -5 + (V_a - V_b)/10 + (V_c - V_b)/5 + V_c/20 = 0$$

$$\text{Super-Node internal Nodes' voltage relationship} \rightarrow V_a - V_c = 3I_x$$

$$\text{Control source variable equation} \rightarrow I_x = (V_a - V_b)/10$$

$$\text{Node "b" KCL} \rightarrow (V_b - V_a)/10 + (V_b - V_c)/5 + V_b/15 = 0$$

Simplify the 4 equations:

$$2V_a - 6V_b + 5V_c = 100$$

$$-3I_x + V_a - V_c = 0$$

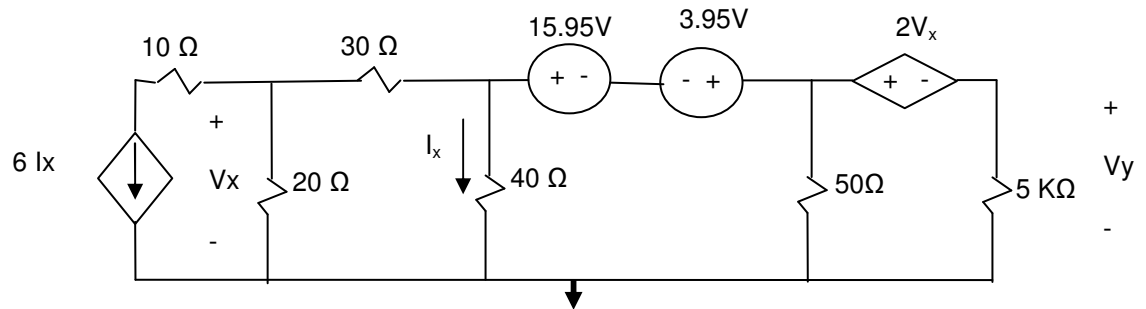
$$10I_x - V_a + V_b = 0$$

$$-3V_a + 11V_b - 6V_c = 0$$

Solve the system of four equations and four unknowns:

$$I_x = 1.1 \text{ A}; V_a = 50.6 \text{ V}; V_b = 39.6 \text{ V}; V_c = 47.2 \text{ V} \rightarrow V_x = V_c = 47.2 \text{ V}$$

- Example 3 – Super Node Example
Find V_y using Node Voltage Method:



Solution:

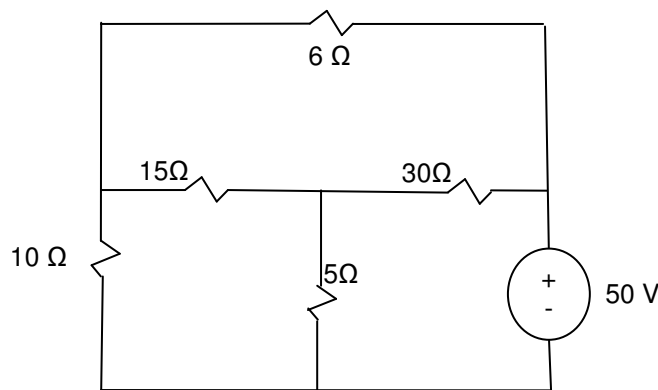
4.3 Mesh-Current Method (based on KVL)

Mesh-Current Method is a formalized and systematic approach to the application of KVL to circuit analysis. These techniques can also be applied to complex circuits since they represent a systematic approach: The following steps describe Mesh-Current Method:

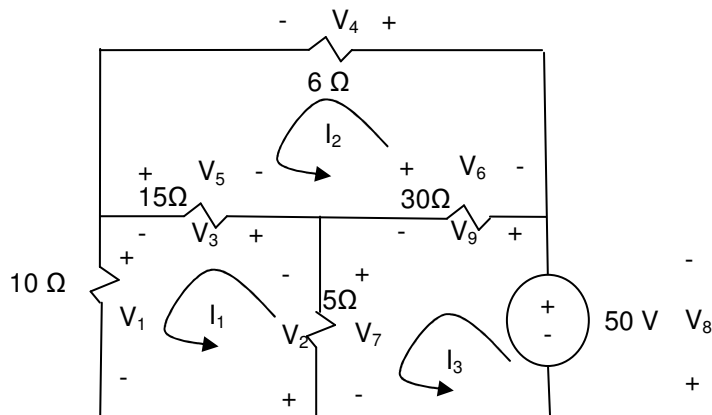
- 1) Redraw the circuit as a planar circuit.
- 2) Identify all meshes in the circuit (Mesh is a loop that does not include other loops) and “m” is the number of meshes.
- 3) Select mesh currents in the same direction (Clockwise or Counter-Clockwise), be consistent throughout the circuit!
- 4) Mesh current exists on the perimeter of a mesh; current in the shared branches is the algebraic sum of the adjoining mesh currents.
- 5) Apply KVL to each of the “m” meshes identified and write the resulting “m” independent equations in term of mesh currents.

❖ Example - Mesh-Current Method

For the circuit shown below, use the Mesh-Current method to find the Power delivered by 50 V supply.



Three meshes have been identified. For the identified meshes, we have selected mesh currents I_1 , I_2 & I_3 with Counter-Clockwise direction. In this case, the voltages are also shown for their educational value.



Now write the KVL equations (algebraic sum of voltages around the mesh and set it to zero) in term of mesh currents for each of the three meshes. It is very important to notice that voltages are marked such that mesh current flows into the + terminals. It is recommended that the reader independently write the equations and then compare them with the equations below. Special attention should be accorded to the signs of the terms in each equation:

$$\begin{aligned}\text{Mesh 1 KVL} &\rightarrow V_1 + V_2 + V_3 = 0 \rightarrow 10I_1 + 5(I_1 - I_3) + 15(I_1 - I_2) = 0 \\ \text{Mesh 2 KVL} &\rightarrow V_4 + V_5 + V_6 = 0 \rightarrow 6I_2 + 15(I_2 - I_1) + 30(I_2 - I_3) = 0 \\ \text{Mesh 3 KVL} &\rightarrow V_7 + V_8 + V_9 = 0 \rightarrow 5(I_3 - I_1) + (-50) + 30(I_3 - I_2) = 0\end{aligned}$$

Simplify and rewrite the equations:

$$\begin{aligned}30I_1 - 15I_2 - 5I_3 &= 0 \\ -15I_1 + 51I_2 - 30I_3 &= 0 \\ -5I_1 - 30I_2 + 35I_3 &= 50\end{aligned}$$

$$\begin{aligned}\text{Solve the equations} &\rightarrow I_1 = 2.7 \text{ A}; I_2 = 3.8 \text{ A}; I_3 = 5.1 \text{ A}; \\ P = V * I &= -(50)(5.1) = -255 \text{ W}\end{aligned}$$

❖ Mesh-Current Method Special Cases

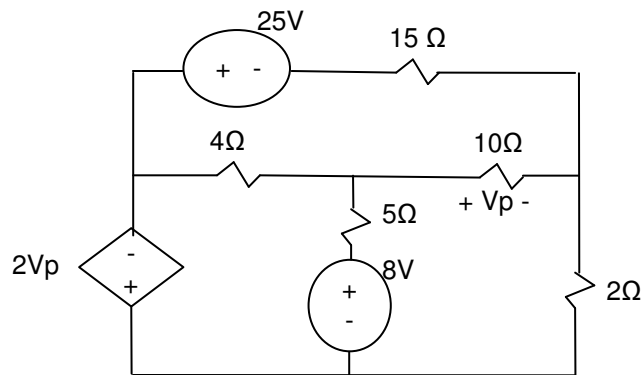
This section lists special cases that are encountered in analysis of circuits using Mesh-Current Method

➤ Special Case 1 – Circuits with Dependent Source

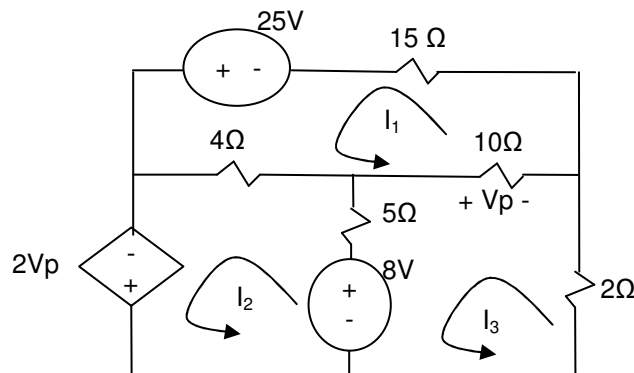
When the circuit under analysis includes a dependent source, the analysis requires an additional variable which is the current or voltage controlling the dependent source.

The analysis step for this type of circuit is similar to Mesh-Current Method described earlier with the additional requirement to write an equation to relate the element voltage to the dependent source variable.

For example, the following circuit contains a dependent voltage source:



Application of Mesh-Current Method is the same as discussed earlier with the addition of a dependent voltage source's variable in-terms of mesh currents..



Write KVL equations for each mesh in terms of the mesh current:

$$\text{Mesh 1 KVL} \rightarrow 4(I_1 - I_2) + 10(I_1 - I_3) + 15I_1 - 25 = 0 \rightarrow 29I_1 - 4I_2 - 10I_3 = 25$$

$$\text{Mesh 2 KVL} \rightarrow -2V_p - 8 + 5(I_2 - I_3) + 4(I_2 - I_1) = 0 \rightarrow -2V_p - 4I_1 + 9I_2 - 5I_3 = 8$$

$$\text{Mesh 3 KVL} \rightarrow +8 + 2I_3 + 10(I_3 - I_1) + 5(I_3 - I_2) = 0 \rightarrow -10I_1 - 5I_2 + 17I_3 = -8$$

There are 3 equations but we have 4 unknowns, so we use the definition of a dependent source to write another equation:

$$V_p = 10(I_1 - I_3) \rightarrow V_p - 10I_1 + 10I_3 = 0$$

So here is the resulting system of 4 equations and 4 unknowns:

$$29I_1 - 4I_2 - 10I_3 = 25$$

$$-2V_p - 4I_1 + 9I_2 - 5I_3 = 8$$

$$-10I_1 - 5I_2 + 17I_3 = -8$$

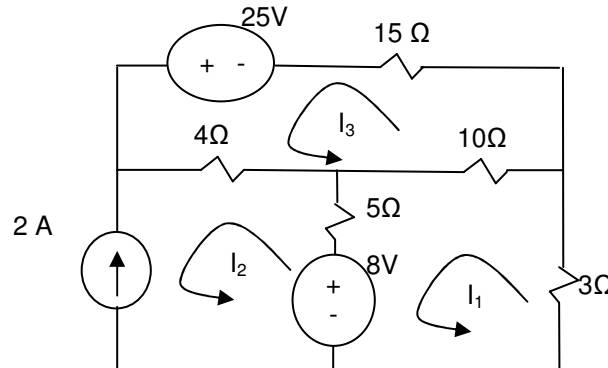
$$V_p - 10I_1 + 10I_3 = 0$$

Solve the system $\rightarrow V_p = 2.9 \text{ V}; I_1 = 1.8 \text{ A}; I_2 = 3.2 \text{ A}; I_3 = 1.6 \text{ A};$

➤ Special Case 2 – Circuits with Current Source

When the circuit under analysis includes dependent or independent current source that is not shared between meshes, the analysis becomes easier since the current in the mesh with the current source is known. This means that there is no need for a KVL equation for the mesh.

For example, the following circuit contains an independent source:



Applying Mesh-Current Method is the same as discussed earlier with the only difference being that the KVL equation for I_2 is not required since $I_2 = -2\text{A}$. So instead of three equations, we need to solve only two equations to find the mesh currents:

$$\text{Mesh 1 KVL} \rightarrow 5(I_1 - I_2) + 8 + 3I_1 + 10(I_1 - I_3) = 0 \rightarrow$$

$$\text{Mesh 2 current is given} \rightarrow I_2 = -2\text{A}$$

$$\text{Mesh 3 KVL} \rightarrow -25 + 4(I_3 - I_2) + 10(I_3 - I_1) + 15I_3 = 0$$

Simplify and plug in I_2

$$18I_1 - 10I_3 = 18$$

$$-10I_1 + 29I_3 = 17$$

Solve the equations $\rightarrow I_1 = 2.6 \text{ A}; I_3 = 1.15 \text{ A};$

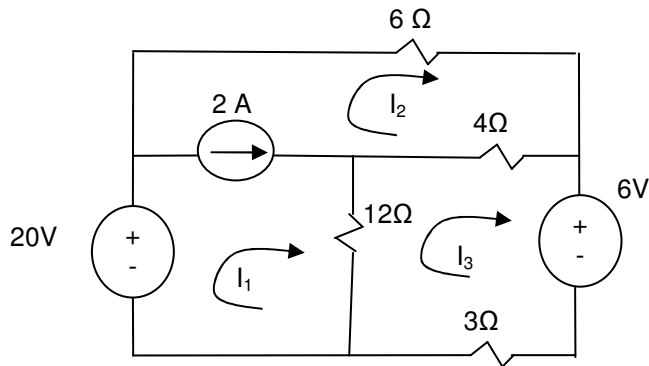
➤ Special Case 3 – Super-Mesh

When a circuit contains a current source between two meshes, we can simplify the analysis by creating a Super-Mesh. A Super-Mesh is a mesh created by combining the two meshes that have the shared current source while avoiding the branch with the current source.

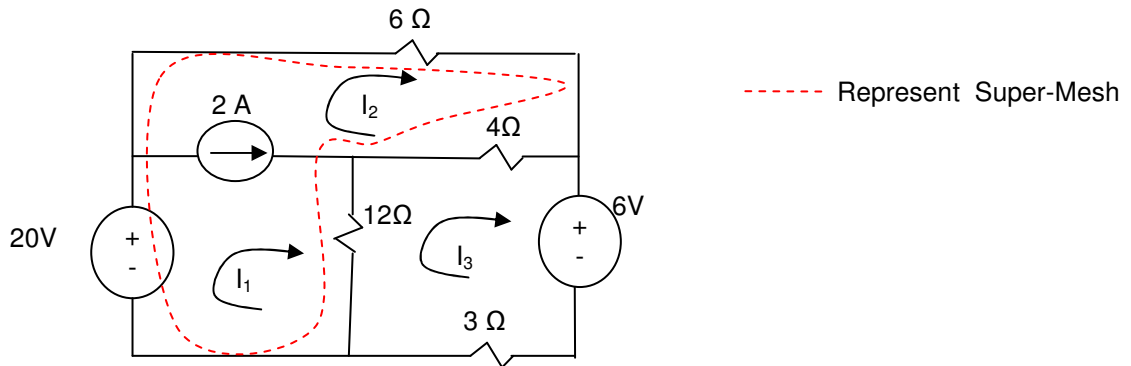
The Super-Mesh concept may be applied to circuits with dependent and independent voltage sources. The Current-Mesh steps apply to circuits with Super-Mesh with the additional requirements to:

- 1) Identify meshes with shared current source and combine them into a Super-Mesh
- 2) Write one Super-Mesh KVL equation
- 3) Write equation relating the current of the two combined meshes

For example in the following in the following circuit, the 2A current source is shared between mesh 1 and 2.



Therefore, mesh 1 and 2 can be combined into a Super-Mesh as shown below:



For the above circuit, equations for Super-Mesh, current in the shared branch and mesh 3 can be written:

$$\text{Super-Mesh KVL} \rightarrow -20 + 6I_2 + 4(I_2 - I_3) + 12(I_1 - I_3) = 0$$

$$\text{Current source branch} \rightarrow 2 = I_1 - I_2$$

$$\text{Mesh 3} \rightarrow 6 + 3I_3 + 12(I_3 - I_1) + 4(I_3 - I_2) = 0$$

Simplify the equations:

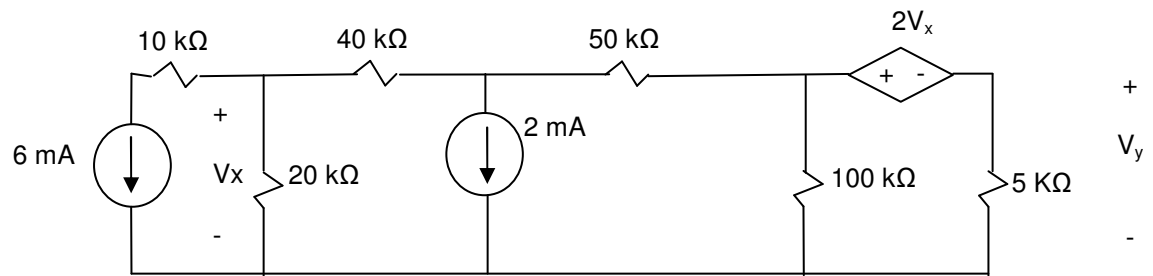
$$12I_1 + 10I_2 - 16I_3 = 20$$

$$I_1 - I_2 = 2$$

$$-12I_1 - 4I_2 + 19I_3 = -6$$

Solve the system of equation $\rightarrow I_1 = 3.3 \text{ A}; I_2 = 1.3 \text{ A}; I_3 = 2.0 \text{ A};$

- ❖ Examples – Application of General Mesh-Current Method
Find V_y in the following circuit using Mesh-Current Method.



➤ Solution:

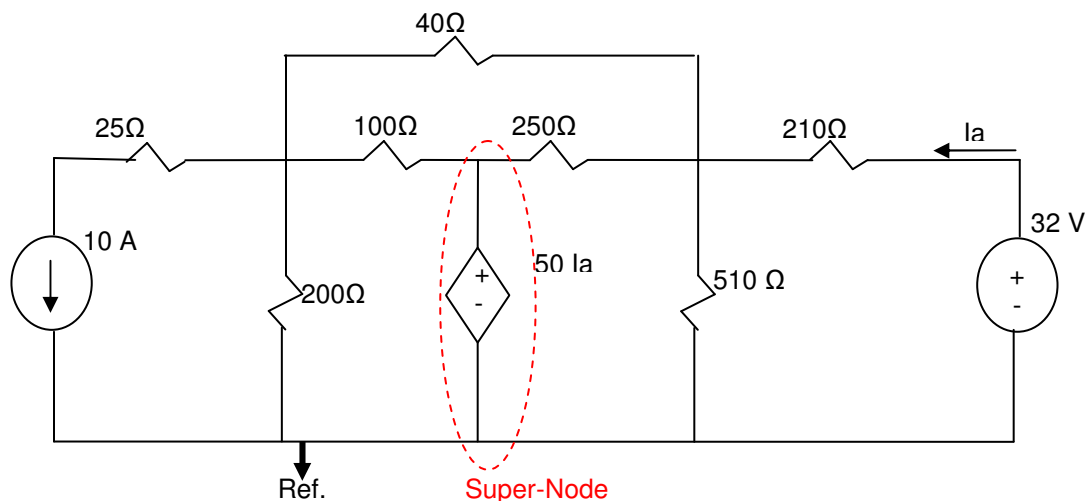
4.4. Node-Voltage and Mesh-Current Comparison

The Node-Voltage and Mesh-Current Methods allow a systematic approach to circuit analysis as shown in previous section. Although both methods are applicable to all planar circuits, for some circuits one method is more efficient and requires less equations and therefore calculation. So spending some time to consider the best approach upfront may save time in the analysis.

Typical factors to consider when deciding between Node-Voltage Method and Mesh-Current Method include:

- Number of nodes and meshes.
 - 1) if $(\# \text{node} - 1) < \# \text{mesh} \rightarrow$ use Node-Voltage Method
 - 2) if $(\# \text{node} - 1) > \# \text{mesh} \rightarrow$ use Mesh-Current Method
 - 3) if $(\# \text{node} - 1) = \# \text{mesh} \rightarrow$ Look for other factors to decide
- Presence of Super-Node or Super-Mesh
 - 1) If Super-Node is present \rightarrow use Node-Voltage Method
 - 2) If Super-Mesh is present \rightarrow use Mesh-Current Method
- Focus of analysis and the unknown parameter.

- ❖ Example – Node-Voltage vs. Mesh-Current
Which of Node-Voltage or Mesh-Current Method is more efficient approach to analyzing the following circuit?

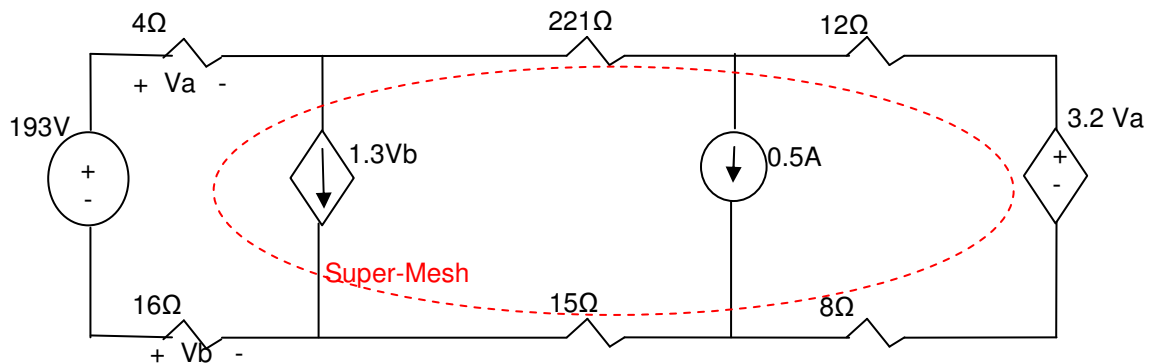


The above circuit has 4 nodes which require 3 equations. There is also opportunity to take advantage of Super-Node which would reduce the number of equation to 2..

The above circuit has 5 meshes which means we have to write 5 Mesh-Current equations.

Based on the fact that node-voltage requires lower number of equations, the recommendation is to use Node-Voltage method to simplify the analysis.

- ❖ Example – Node-Voltage vs. Mesh-Current
Does Node-Voltage or Mesh-Current approach requires less effort to analyze the following circuit?



The above circuit has 4 nodes therefore requires 3 equations with Node-Voltage Method.

The above circuit has 3 meshes which would require 3 equations with Mesh-Current Method. But considering the current sources and existence of Super-Mesh, we only require one equation.

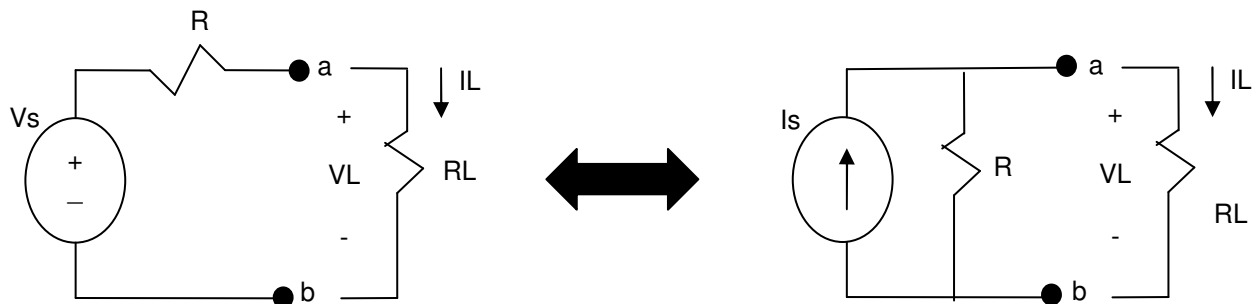
Based on the fact that node-voltage requires higher number of equation, the recommendation is to use Mesh-Current Method for more efficient analysis.

4.5. Source Transformation

Source Transformation is another circuit simplification and analysis technique. In electrical circuit analysis, as long as the voltage and current are maintained, an equivalent circuit may be used.

Source Transformation is a technique which allows current source to be replaced with a voltage source and visa-a-versa.

As shown below, the following voltage source configuration and current source configuration are equivalent at terminals a and b:



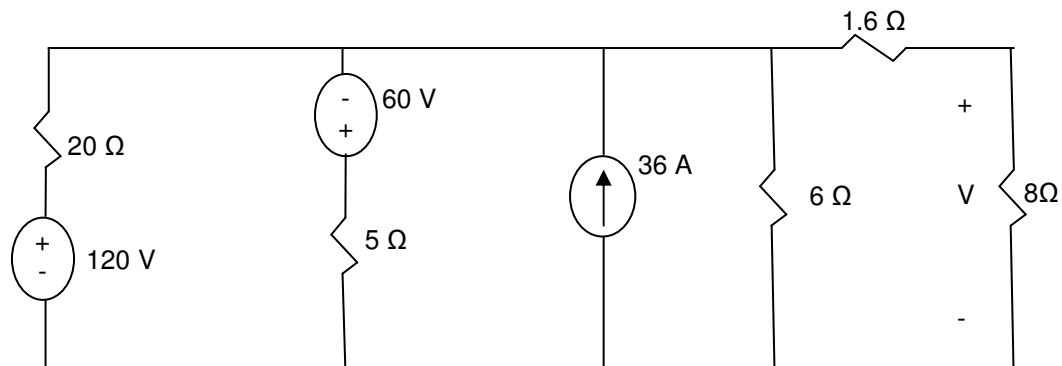
Valid transformation requires that I_L & V_L has to be the Same. For the above circuits:

- Voltage at terminal ab is V_L for both circuits
- Current at terminal ab is I_L for both circuits

The only relationship required to transform from one form to another is: **$I_s = V_s / R$**

❖ Example - Source Transformation

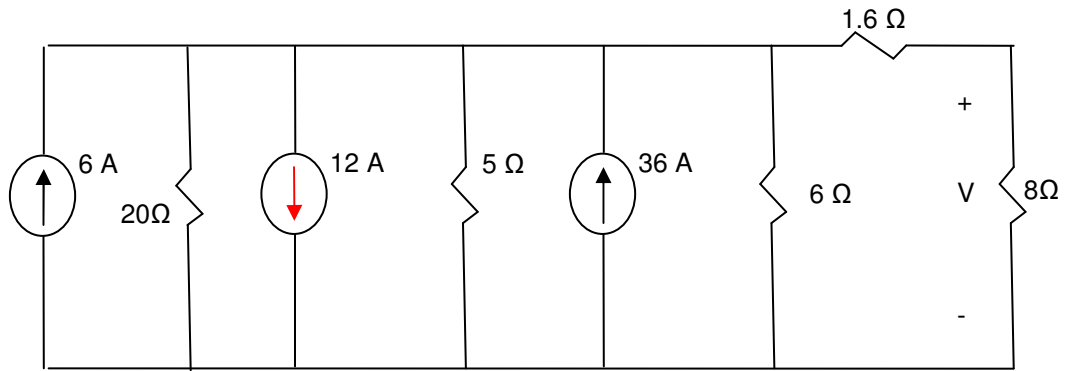
User source transformation to find the voltage V in the circuit shown below:



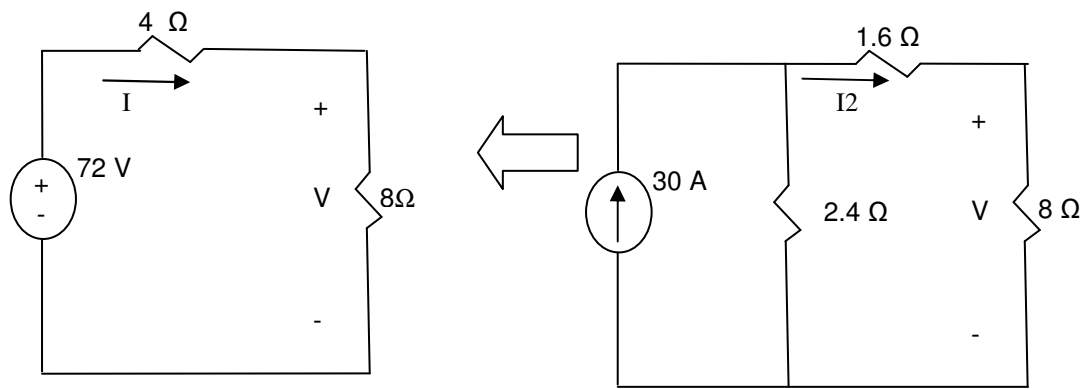
Transformations:

- (1) 120 V & 20 Ω Series $\rightarrow 120/20 = 6$ A and 20 Ω parallel
- (2) 60 V & 5 Ω Series $\rightarrow 60/5 = 12$ A and 5 Ω parallel

There fore



Or

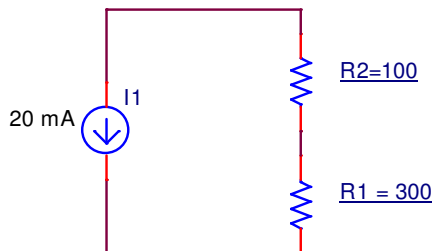


$$I = 72 / (4 + 8) = 6 \text{ A}$$

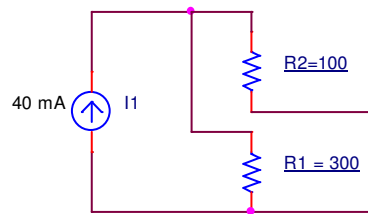
$$V = 8 \times 6 = 48 \text{ V}$$

- ❖ Example - Source Transformation
Draw the equivalent circuit with respect to current source terminals using only resistor and voltage source for the following circuits:

a)



b)

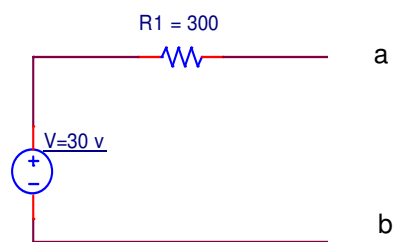


❖ Solution:

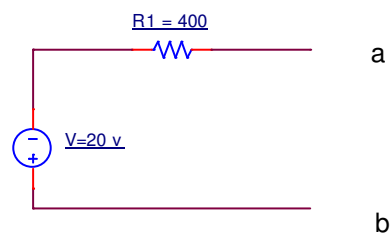
Example - Source Transformation

Draw the equivalent circuit with respect to a and b terminals using only resistor and current source for the following circuits:

a)



b)

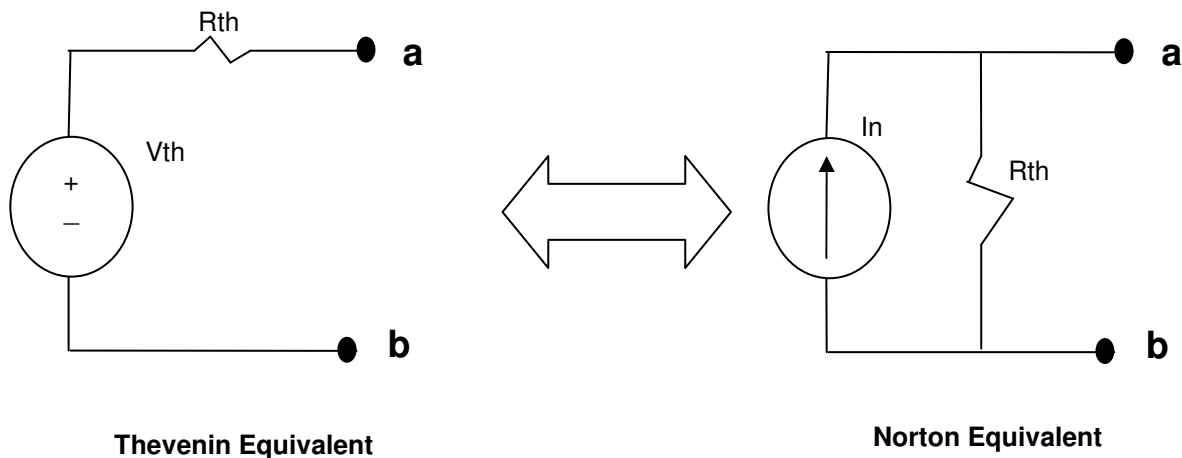


➤ Solution:

4.6. Thevenin & Norton Equivalents

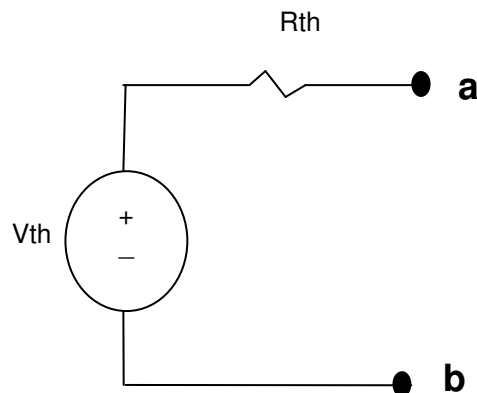
In many cases, it is useful to replace the circuit under analysis with a simpler circuit where its voltage and current at the terminals behave the same as the original circuit. Thevenin and Norton Equivalents enable us to come up with equivalent circuit consisting of one source and one resistor that is equivalent to the original circuit.

The rest of this section outlines the process of finding a Thevenin or Norton Equivalent for any ideal circuit. Thevenin equivalent consists of a voltage source and resistor while the Norton equivalent consists of a current source and resistor as shown below. Further, the two forms are equivalent to each other.



❖ Thevenin Equivalent (Voltage Source & Resistor)

Thevenin Equivalent of a circuit is represented by only a voltage source referred to as the V_{th} or open connection voltage (V_{oc}) and an equivalent or Thevenin resistor (R_{th}).



Definitions:

- Thevenin or Open Circuit Voltage ($V_{th} = V_{oc}$) is voltage at terminal a and b when Current through terminal a and b is zero (open across a & b).
- Norton or Short Circuit Current ($I_n = I_{sc}$) is the current through terminals a & b when Voltage across a and b is zero (Short across a & b)
- Thevenin or equivalent resistor (R_{th}) is the ratio of Open Circuit voltage and Short Circuit current:

$$R_{th} = V_{oc} / I_{sc}$$

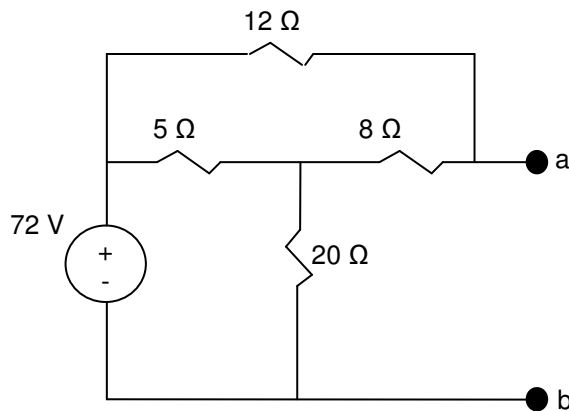
❖ Thevenin equivalent using Standard Method

Thevenin equivalent of circuit is found with respect to a given pair of terminals since the Thevenin equivalent will be different depending on the terminals selected. Once the terminals and the circuit have been provided, calculate two of the three variables R_{th} , V_{oc} or I_{sc} to find the equivalent circuit. With two of variables, the third can be calculated using the Ohm's law. Here are the steps for the standard method:

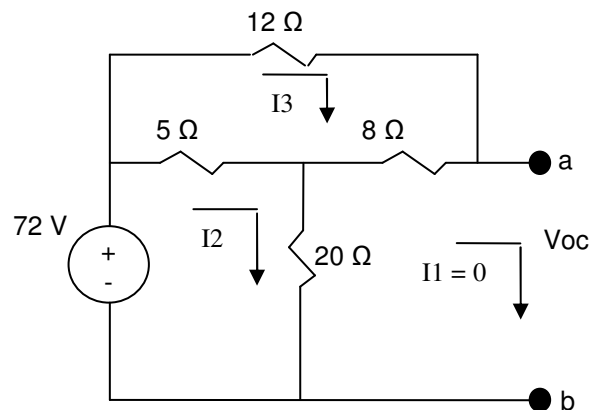
- (1) Find the current across the terminals when the terminals are shorted (connected). This is the short circuit current, $I_{sc}=I_n$.
- (2) Find the voltage across the terminals when the terminals are open. This is the open circuit or Thevenin $V_{oc}=V_{th}$.
- (3) Use Ohm's law to find $R_{th} = V_{oc}/I_{sc}$
- (4) Draw the Thevenin equivalent circuit.

➤ Example - Apply the Standard Method

Find the Thevenin Equivalent for the following circuit at terminals ab.



- (1) Open terminals ab and find the voltage at terminals ab ($V_{oc} = V_{th}$)



Mesh 1 → terminals ab are open → $I_1 = 0$

Mesh 2 → $-72 + 5(I_2 - I_3) + 20(I_2) = 0$

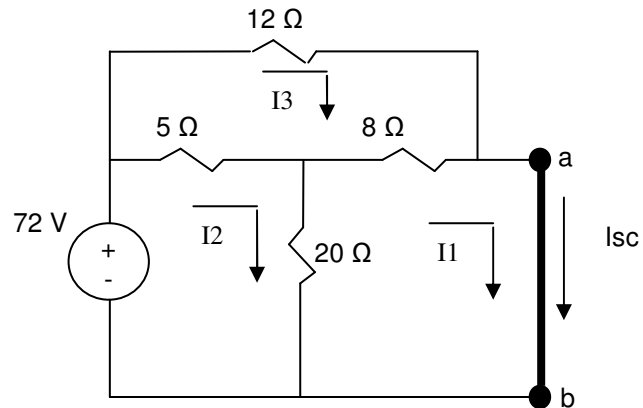
Mesh 3 → $12 I_3 + 8(I_3) + 5(I_3 - I_2) = 0$

Simplify

$$\begin{aligned} 25I_2 - 5I_3 &= 72 \\ -5I_2 + 25I_3 &= 0 \end{aligned}$$

$$I_2 = 3 \text{ A}; I_3 = 3/5 \text{ A} \rightarrow V_{oc} = V_{th} = 8 \cdot 3/5 + 20 \cdot 3 = 64.8 \text{ V}$$

(2) Short the circuit at terminals ab and find I_{sc}



Applying Mesh-Current method

$$\begin{aligned} \text{Mesh 1} &\rightarrow 20(I_1 - I_2) + 8(I_1 - I_3) = 0 \\ \text{Mesh 2} &\rightarrow -72 + 5(I_2 - I_3) + 20(I_2 - I_1) = 0 \\ \text{Mesh 3} &\rightarrow 12I_3 + 8(I_3 - I_1) + 5(I_3 - I_2) = 0 \end{aligned}$$

Simplify the equations

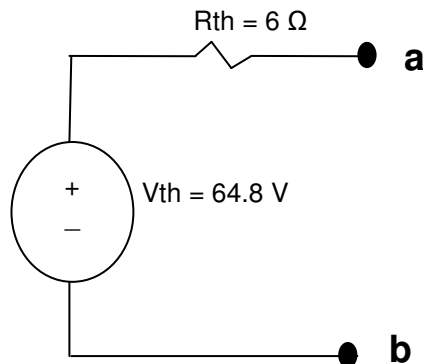
$$\begin{aligned} 28I_1 - 20I_2 - 8I_3 &= 0 \\ -20I_1 + 25I_2 - 5I_3 &= 72 \\ -8I_1 - 5I_2 + 25I_3 &= 0 \end{aligned}$$

$$\begin{aligned} \text{Solve} &\rightarrow I_1 = 10.8 \text{ A}; I_2 = 12.7 \text{ A}; I_3 = 6.0 \text{ A} \\ I_{sc} &= I_1 = 10.8 \end{aligned}$$

(3) Calculated R_{th}

$$\text{We have } R_{th} = V_{oc}/I_{sc} = 64.8/10.8 = 6 \Omega$$

(4) Thevenin Equivalent Circuit



❖ Thevenin Equivalent – Special Cases

In some cases, it may be easier (less calculations) to use an alternative method to find the Circuits Thevenin Equivalents.

➤ Special Case 1 – Circuits only contains Independent sources

For this type of circuit, it is typically easier to find Thevenin equivalent by applying the following steps:

- 1) Find Open circuit voltage at the terminals, $V_{oc}=V_{th}$, same as in the Standard method.
- 2) Deactivate all independent current sources
An independent current source is deactivated by replacing it with an open circuits ($I=0$)
- 3) Deactivate all independent voltage
An independent voltage source is deactivated by replacing it with a short circuit ($V=0$)
- 4) Calculate R_{eq} for the resulting circuit which is the same as R_{th}
- 5) Draw the Thevenin equivalent using the calculated V_{th} and R_{th}

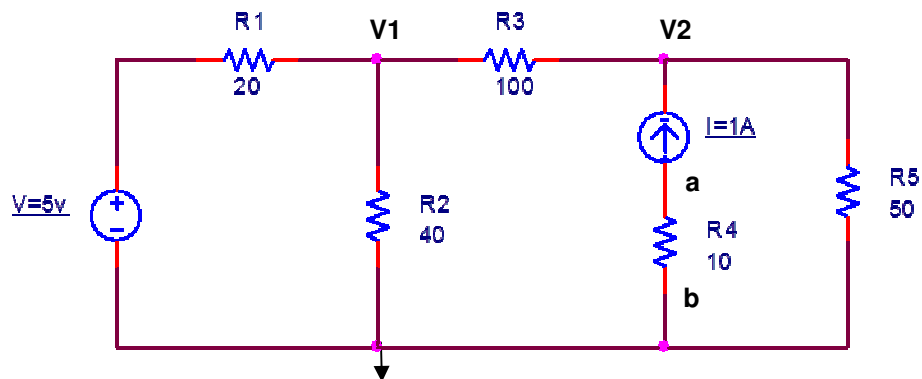
➤ Special Case 2– Circuits with dependent and independent sources

For this type of circuit, it is sometime easier to find Thevenin equivalent by applying the following steps:

- 1) Find Open circuit voltage at the terminals, $V_{oc}=V_{th}$, same as in the Standard method.
- 2) Deactivate all independent current sources
An independent current source is deactivated by replacing it with an open circuits ($I=0$)
- 3) Deactivate all independent voltage sources
An independent voltage source is deactivated by replacing it with a short circuit ($V=0$)
- 4) Use either test voltage source or current source
 - (1) Calculate test current while applying test voltage at terminals a and b.
 - OR
 - (2) Calculate test voltage while applying test current at terminals a and b.
- 5) $R_{th} = V_{test}/I_{test}$
- 6) Draw the Thevenin equivalent using the calculated V_{th} and R_{th}

➤ Example – Thevenin Equivalent

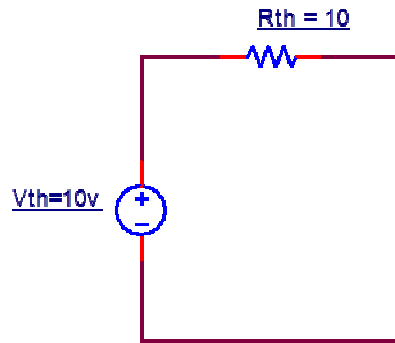
Find the Thevenin equivalent of the following circuit with respect to terminal ab (across R_4):



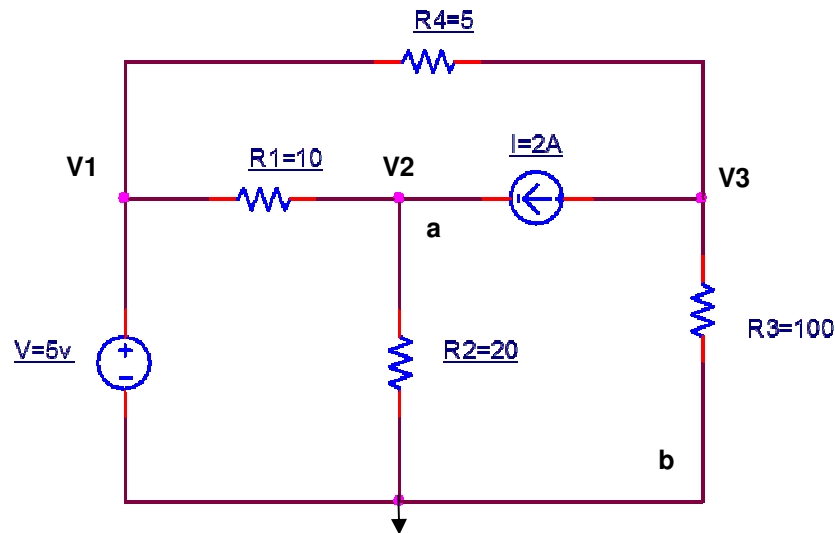
Solution:

1. Find $V_{oc}=V_{th}$ with terminals ab open
see that 1 A is the current in that branch and resistance is 10 Ω therefore
 $V_{ab} = V_{th} = (-1)(10) = -10 \text{ V}$

2. Find $I_N = I_{sc}$ with terminals ab short circuited
 $I_{sc} = -1A$
3. Calculate $R_{th} = V_{th} / I_{sc}$
 $R_{th} = -10/-1 = 10\Omega$
4. Draw the Thevenin equivalent Circuit



- Example – Thevenin Equivalent
 Find the Thevenin equivalent of the following circuit with respect to terminal ab:



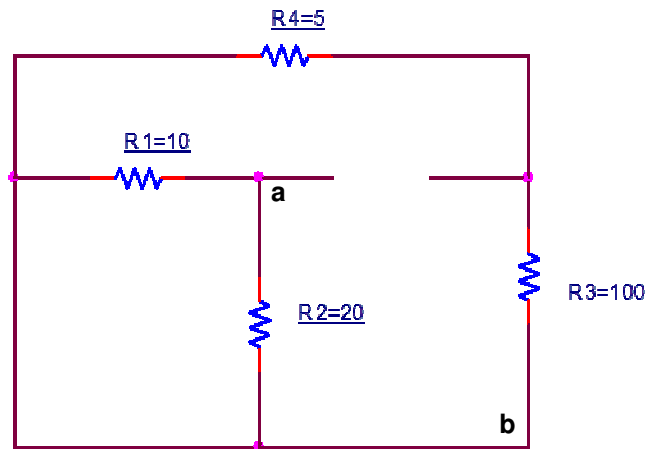
Solution:

Since this circuit only has independent source, we are able to use the special case 2 to find Thevenin equivalent.

1. Find $V_{oc}=V_{th}$ with terminals ab open
 KCL at V1 $\rightarrow V1 = 5V$
 KCL at V2 $\rightarrow (V2 - V1) / 10 + V2/20 - 2 = 0$
 KCL at V3 $\rightarrow (V3 - V1)/5 + 2 + V3/100 = 0$

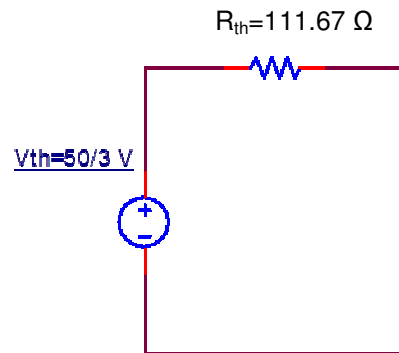
$$V_{th} = V_{ab} = V2 = 50/3 = 16.7 V$$

2. Find R_{eq} with respect to terminal a and b when sources have been deactivated (open current source and short voltage source)

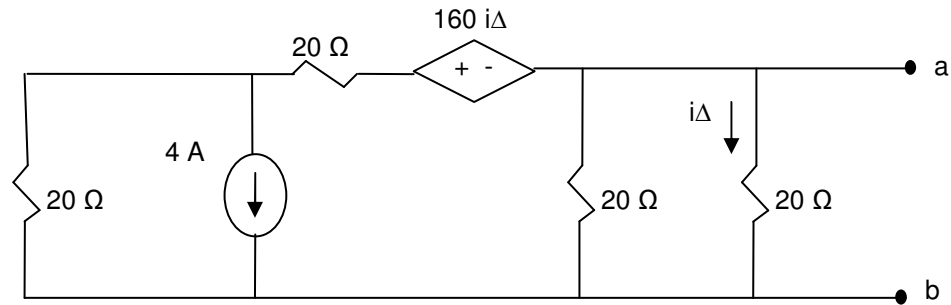


$$R_{th} = R_{ab} = (10 \parallel 20) + 5 + 100 = 111.67\ \Omega$$

3. Draw the Thevenin equivalent Circuit



- Example – Thevenin Equivalent
Find the Thevenin equivalent of the following circuit with respect to terminal ab:



Solution:

1. Find $V_{oc} = V_{th}$ with terminals ab open

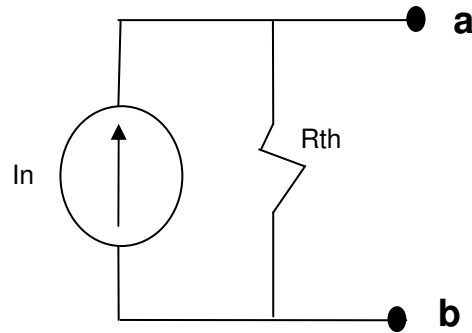
2. Find $I_N = I_{sc}$ with terminals ab short circuited

3. Calculate $R_{th} = V_{th} / I_{sc}$

4. Draw the Thevenin equivalent Circuit

❖ Norton Equivalent (Current Source & Resistor)

Norton and Thevenin are source transformation of each other. The method is very similar to Thevenin Equivalent with the only difference is that the equivalent circuit consists of a current source, I_{sc} , and resistor, R_{th} . The Norton Equivalent circuit is shown below



Definitions:

- Thevenin or open circuit voltage ($V_{th} = V_{oc}$) is voltage at terminal a and b when Current through terminal a and b is zero (open across a & b).
- Norton or Short Circuit Current ($I_n = I_{sc}$) is the current through terminals a & b when Voltage across a and b is zero (short across a & b)
- Thevenin or equivalent resistance (R_{th}) is the ratio of open circuit voltage and short circuit current:

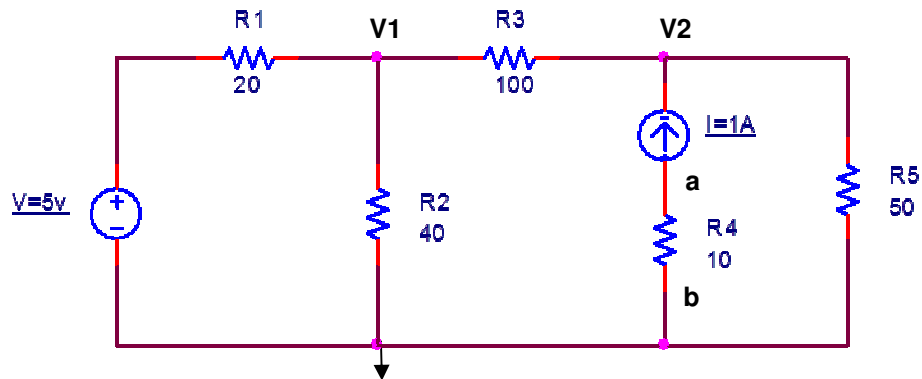
$$R_{th} = V_{oc} / I_{sc}$$

❖ Norton equivalent using Standard Method

Norton equivalent of circuit is found with respect to a given pair of terminals similar to Thevenin equivalent. Once the terminals and circuit has been provided, calculate two of the three variables R_{th} , V_{oc} or I_{sc} to find the equivalent circuit. With two of variables, the third can be calculated using the Ohm's law.. Here are the steps for the standard method:

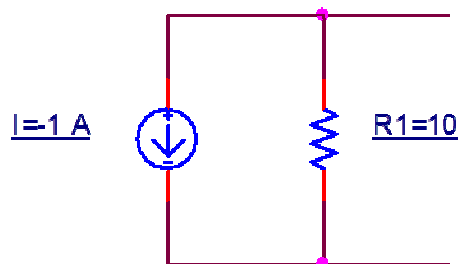
- (2) Find the current across the terminals when the terminals are shorted (connected). This is the short circuit current or Norton current, $I_{sc} = I_n$.
- (3) Find the voltage across the terminals when the terminals are open. This is the open circuit or Thevenin $V_{oc} = V_{th}$.
- (4) Use Ohm's law to find $R_{th} = V_{oc} / I_{sc}$
- (5) Draw the Norton equivalent circuit.

- Example – Norton Equivalent
Find the Norton equivalent of the following circuit with respect to terminal ab (across R4):

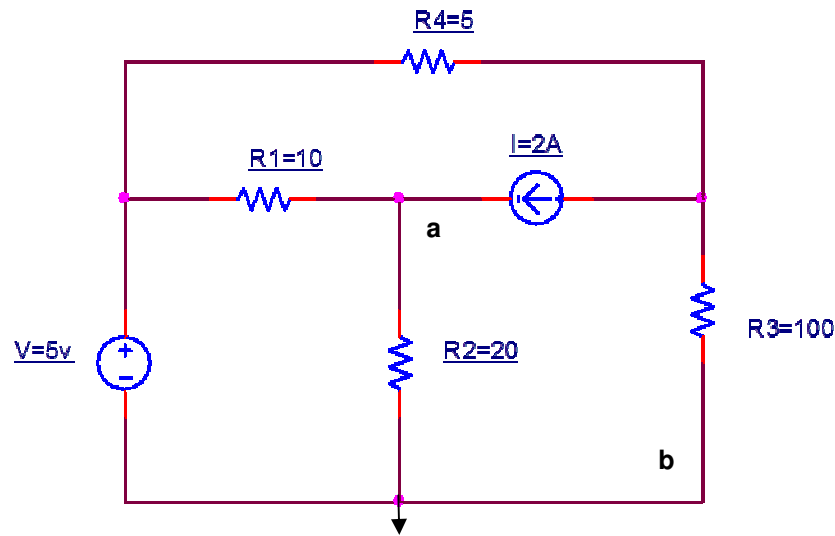


Solution:

1. Find $V_{oc} = V_{th}$ with terminals ab open
see that 1 A is the current in that branch and resistance is $10\ \Omega$ therefore
 $V_{ab} = V_{th} = (-1)(10) = -10\text{ V}$
2. Find $I_N = I_{sc}$ with terminals ab short circuited
 $I_{sc} = -1\text{ A}$
3. Calculate $R_{th} = V_{th} / I_{sc}$
 $R_{th} = -10 / -1 = 10\ \Omega$
4. Draw the Norton equivalent Circuit



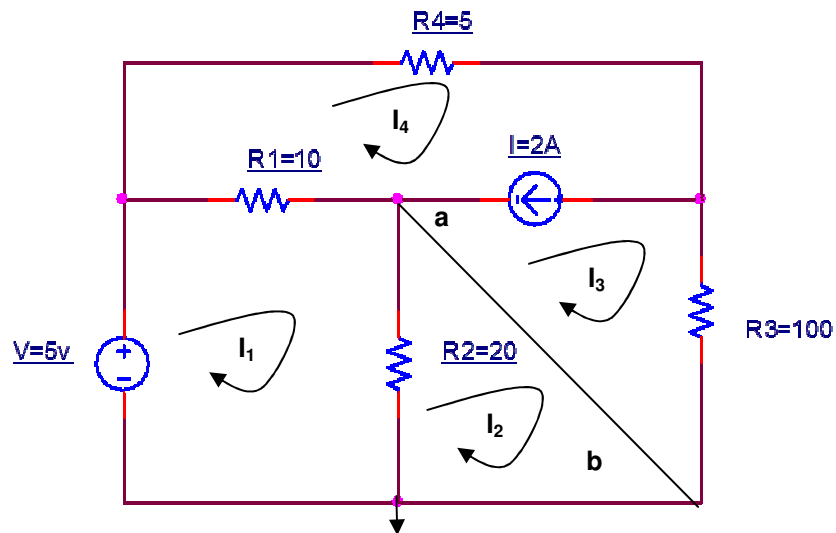
- Example – Norton Equivalent
Find the Norton equivalent of the following circuit with respect to terminal ab:



Solution:

Since this circuit only has independent source, we are able to use the special case 2 to find thevenin equivalent.

1. Find $I_{sc}=I_{ab}$ with terminals ab Shortened



$$\text{KVL at } I_1 \rightarrow -5 + 10(I_1 - I_4) + 20(I_1 - I_2) = 0$$

$$\text{KVL at } I_2 \rightarrow I_2 = 0$$

$$\begin{aligned} \text{KVL at supper mesh } I_3 \text{ \& } I_4 &\rightarrow 5I_4 + 100I_3 + 10(I_4 - I_1) = 0 \\ &\rightarrow I_4 - I_3 = 2 \end{aligned}$$

Simplify:

$$30I_1 - 10I_4 = 5$$

$$10I_1 + 100I_3 - 5I_4 = 0$$

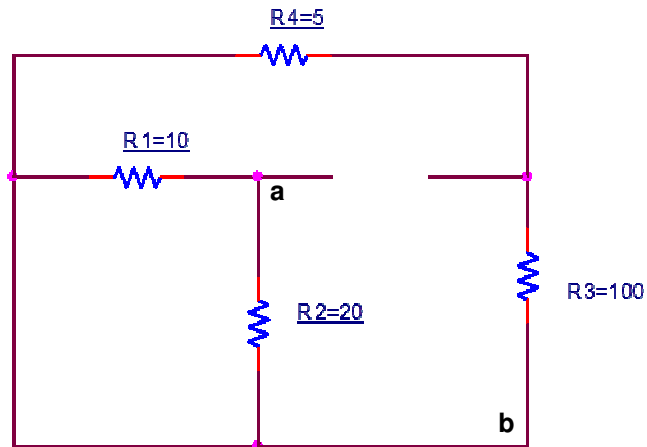
$$I_4 - I_3 = 2$$

Solve to find:

$$I_2 = 0, \quad I_3 = -2.66 \text{ A}$$

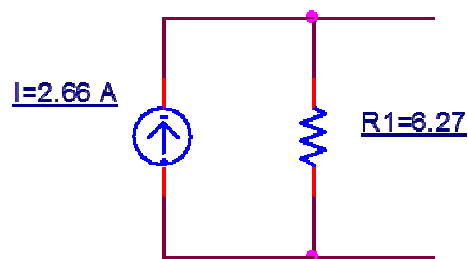
$$I_{sc} = I_{ab} = I_2 - I_3 = 2.66 \text{ A}$$

2. Find R_{eq} with respect to terminal a and b when sources have been deactivated (open current source and short voltage source)

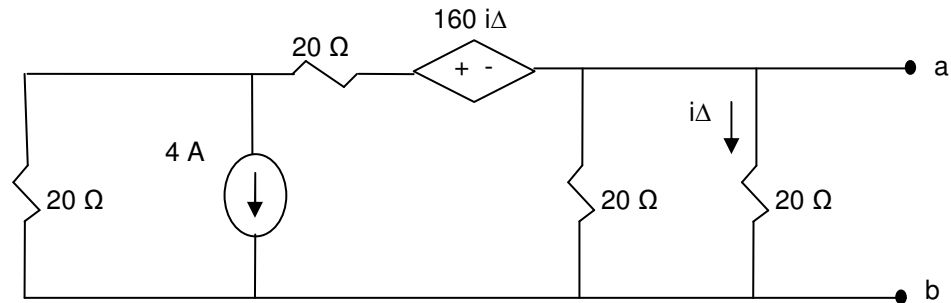


$$R_{th} = R_{ab} = (10 \parallel 20 \parallel (5 + 100)) = 6.27 \Omega$$

3. Draw the Norton equivalent Circuit



- Example – Norton Equivalent
Find the Norton equivalent of the following circuit with respect to terminal ab:



Solution:

1. Find $V_{oc} = V_{th}$ with terminals ab open

2. Find $I_N = I_{sc}$ with terminals ab short circuited

3. Calculate $R_{th} = V_{th} / I_{sc}$

4. Draw the Norton equivalent Circuit

4.7. Delivering Maximum Power

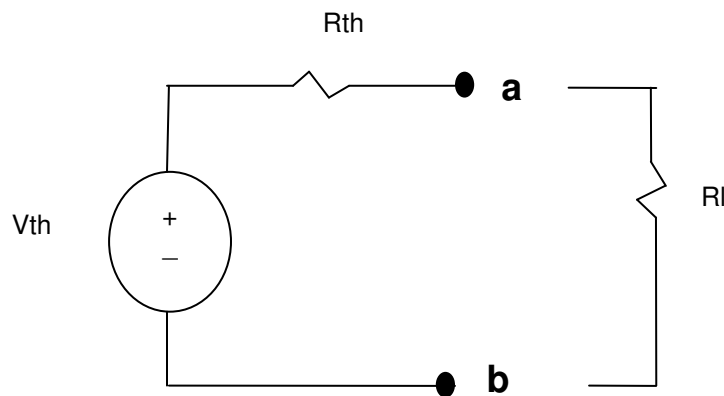
There is a cost associated with electrical power generation therefore producers objective are to deliver the maximum power to the load (user). This is also true when applied to radio, cell phone and other devices. Although, the actual calculation for devices other than resistors and supply is beyond this section, the underlying concept of wanting to maximize power to the load is applicable.

❖ Requirements for delivering maximum Power

The following steps are used to prove the load and generating circuit configurations that delivers the maximum power delivered to the load (user) must meet the following conditions:

$$R_{th} = R_l \Rightarrow p_{\max} = \frac{V_{th}^2}{4R_l} = \frac{V_{th}^2}{4R_{th}}$$

- Find and draw the Thevenin equivalent of the generator circuit and represent the load as R_L



- Calculate the power of the load connected to Thevenin equivalent circuit

$$p = I^2 R_l = \left(\frac{V_{th}}{R_l + R_{th}} \right)^2 R_l$$

To find the maximum power, set the derivative of p with respect to R_L to zero. Note for the following calculation R_L is a variable.

$$\frac{dp}{dR_l} = V_{th}^2 \left[\frac{(R_l + R_{th})^2 - R_l * 2(R_l + R_{th})}{(R_l + R_{th})^4} \right]$$

Derivate must be 0 to find the R_L of P_{\max} . So

$$(R_l + R_{th})^2 - R_l * 2(R_l + R_{th}) = 0$$

or

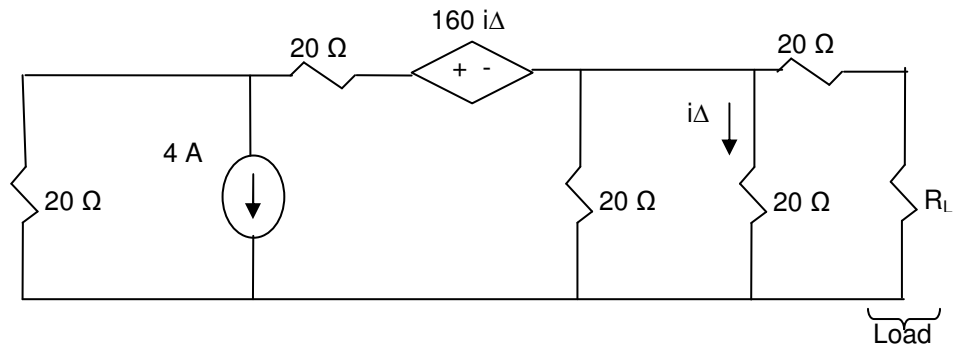
$$(R_l + R_{th})^2 = 2R_l(R_l + R_{th}) \Rightarrow R_l = R_{th}$$

- $R_L = R_{th}$ is the condition for obtaining maximum

$$P_{\max} = \left(\frac{V_{th}}{R_L + R_{Rth}} \right)^2 R_L = \frac{V_{th}^2}{4R_L} = \frac{V_{th}^2}{4R_{th}}$$

❖ Example – Maximum Power

Find the value of R_L in order for maximum power to be delivered to the load:



Solution:

❖ Example – Maximum Power

North Bonneville Dam generates 500 MW at 100 kV. For this system, find load (R_L) such that maximum power is delivered to the load.

Solution:

4.8. Super-Position Principle

Super-Position Principle can only be applied to linear systems. Fortunately, the ideal circuits with only linear elements such as resistor, independent source and dependent source with power of one control variables are linear systems. This means we can apply the Super-Position Principles to simplify, analyze and design these circuits. The Super-Position Principle simply states that the effect of each independent source can be added to find the total response in a linear system.

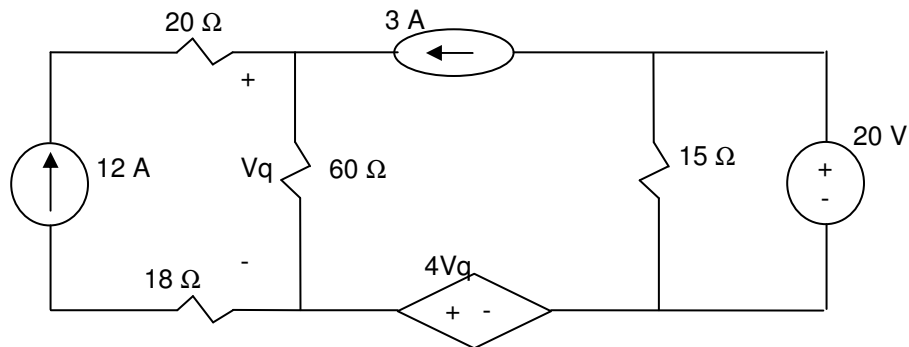
❖ Application of Super-position to Circuits Analysis

Here are steps to apply the Super-Position to Circuits:

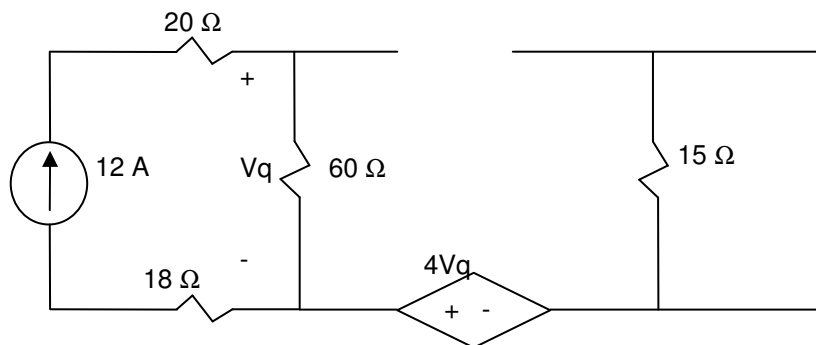
- Deactivate all independent sources (short for voltage source and open for current source)
Note: Dependent sources cannot be deactivated
- Activate one independent source at a time and calculate the response for it
- Repeat the previous step for all the independent sources
- Sum all the individual responses to find the total response

❖ Example – Application of super-position principal

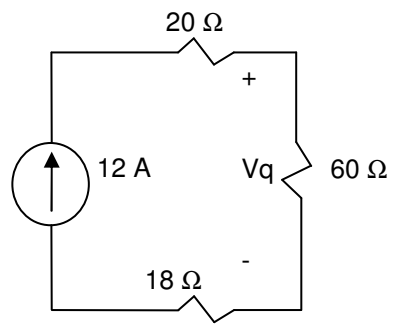
Find V_q in the following Circuit using super position:



- 1) Deactivate all independent sources except 12 A current source

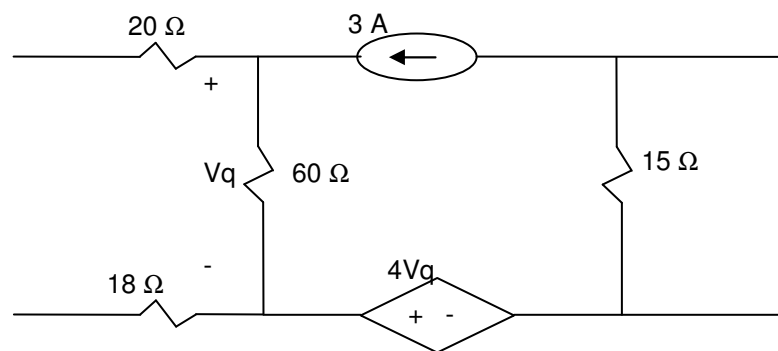


Circuit the simplifies to :

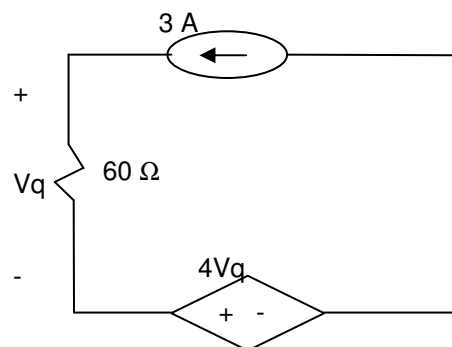


$$V_{q1} = 60 \cdot 12 = 720 \text{ v.}$$

2) Deactivate all independent sources except 3A current source

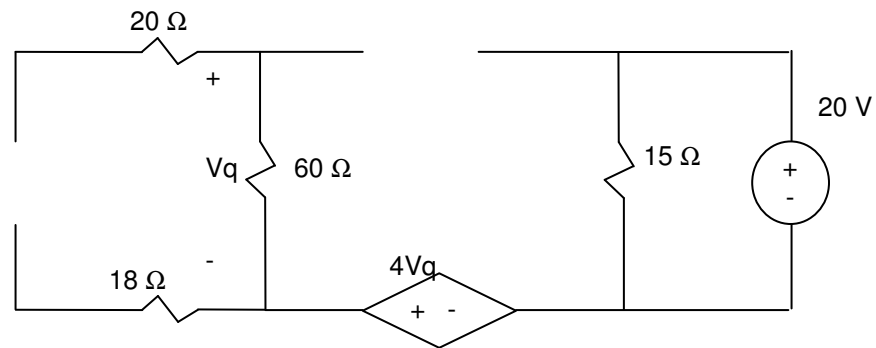


Circuit the simplifies to :



$$V_{q2} = 60 \cdot 3 = 180 \text{ v.}$$

- 3) Deactivate all independent sources except 20 V current source



There are not any closed loops between the independent source and the 60 Ohms resistor therefore $V_{q3} = 0$

- 4) Total value of V_q can be calculated by summing the individual responses:
 $V_q = V_{q1} + V_{q2} + V_{q3} = 720 + 180 + 0 = 900 \text{ V}$

4.9. Sensitivity Analysis

Sensitivity analysis is the process of determining the relative effect of change in one element of the circuit on specific voltage or current in the circuit. The following steps are used to find out sensitivity of voltage or current with respect to an element (resistor) in the circuit:

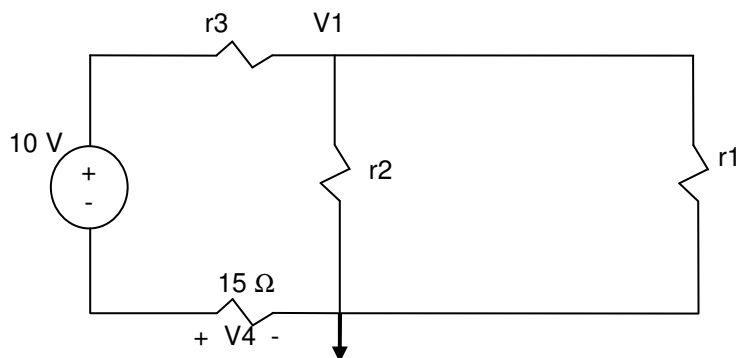
- 5) Write the voltage or current in term of the element value ($V=f(r_1, r_2, \dots)$)
- 6) $dv/dr_1 \quad V/\Omega$ gives us the sensitivity V with respect to r_1

As discussed in the previous section, this course is focused on linear systems (circuits) which mean we can apply the Super-Position Principle to the sensitivity analysis. In other words, calculate sensitivity with respect to individual elements and sum them to calculate the total sensitivity as shown below:

V sensitivity to r_1 and $r_2 = dv/dr_1 + dv/dr_2$

❖ Example - Sensitivity Analysis

What is the sensitivity of V_4 to r_1 , r_2 and r_3 .



Use Node Voltage method to write V_1 in terms of $r_1, r_2, r_3 \rightarrow (V_1 - 10)/(15+r_3) + V_1/r_1 + V_1/r_2 = 0$
 Applying ohms to branch with the voltage source $\rightarrow V_4/15 = (V_1 - 10)/(15+r_3)$

Simplify the first equation $\rightarrow (1/(15+r_3) + 1/r_1 + 1/r_2)V_1 = 10/(15+r_3)$
 From Second equation find V_1 in term of $V_4 \rightarrow V_1 = V_4 (15 + r_3)/15 + 10$

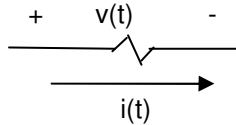
plug in V_1 from 2nd equation to first one \rightarrow

reader is encouraged to complete the example

4.10. Summary

❖ Ohms Law

$$R = \frac{V}{I} \Omega$$



❖ Node Voltage Method (KCL)

$$\left[\sum_{EN} (i_n) \quad \xrightarrow{v = iR} \quad f_n(v_n) \right]$$

- 1) Identify number of Essential Nodes (EN: nodes with more than 2 branches)
- 2) Set of one of the EN to Reference Node ($V_{ref} = 0$)
"Select the EN with most branches"
- 3) For Each of the other ENs write the sum of currents equation in terms of node voltage
 - * Assume current leaving the node is positive
 - * Label Nodes V_1, V_2, V_3, \dots
 - * Write currents in term of V_1, V_2, V_3, \dots
- 4) Solve the system of equations to find V_1, V_2, V_3, \dots

Special Cases:

- a) Voltage Source from Essential Node to Reference Node
- b) Dependent Source
- c) Voltage Source between two Essential Node

❖ Mesh Current Method (KVL)

$$\left[\sum_{Mesh} (v_m) \quad \xrightarrow{R = v / i} \quad f_n(i_n) \right]$$

- 1) Identify number of Meshes (Loops that do not include other loops)
- 2) For Each of Meshes write the sum of voltage equation in terms of current
 - * Label Mesh current (clock wise) I_1, I_2, I_3, \dots
 - * Write voltages in term of I_1, I_2, I_3, \dots
- 3) Solve the system of equations to find I_1, I_2, I_3, \dots

Special Cases:

- 1) Current source in unshared branch
- 2) Dependent Source
- 3) Current source in shared branch

4.11. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 4.

4.12. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 5. Operational Amplifier

Key Concepts and Overview

- ❖ Transistor Overview
- ❖ Operational Amplifier (Op-Amp)
- ❖ Op Amp Applications
- ❖ Common Mode (cm) vs. Differential Mode (dm)
- ❖ Op Amp DC Model
- ❖ Additional Resources

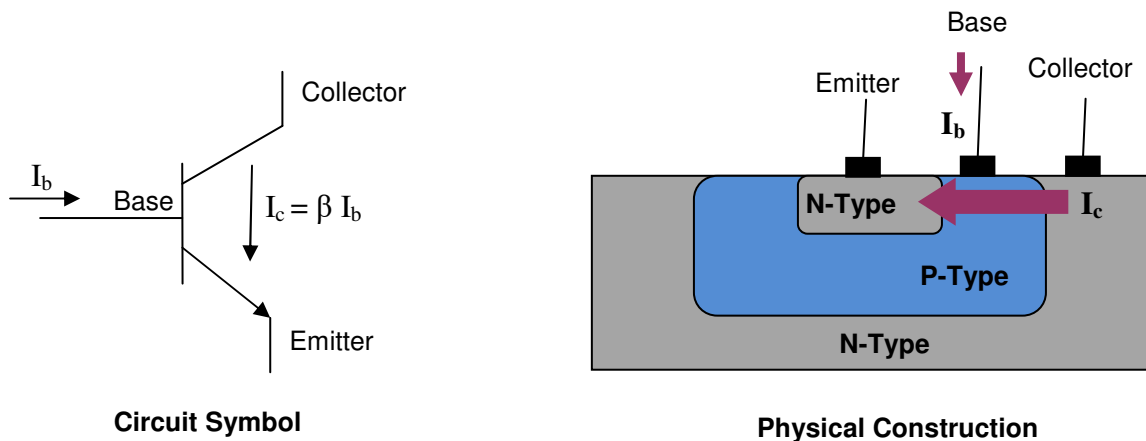
5.1. Transistor Overview

Transistors are fundamental building blocks of electrical and computer systems. A Transistor may be used to amplify current or voltage (as an amplifier), or as a switch. In its operation as an amplifier, transistors are used to build Op Amps and a multitude of other analog devices. In its operation as a switch, a transistor is the building block of all computer and digital systems.

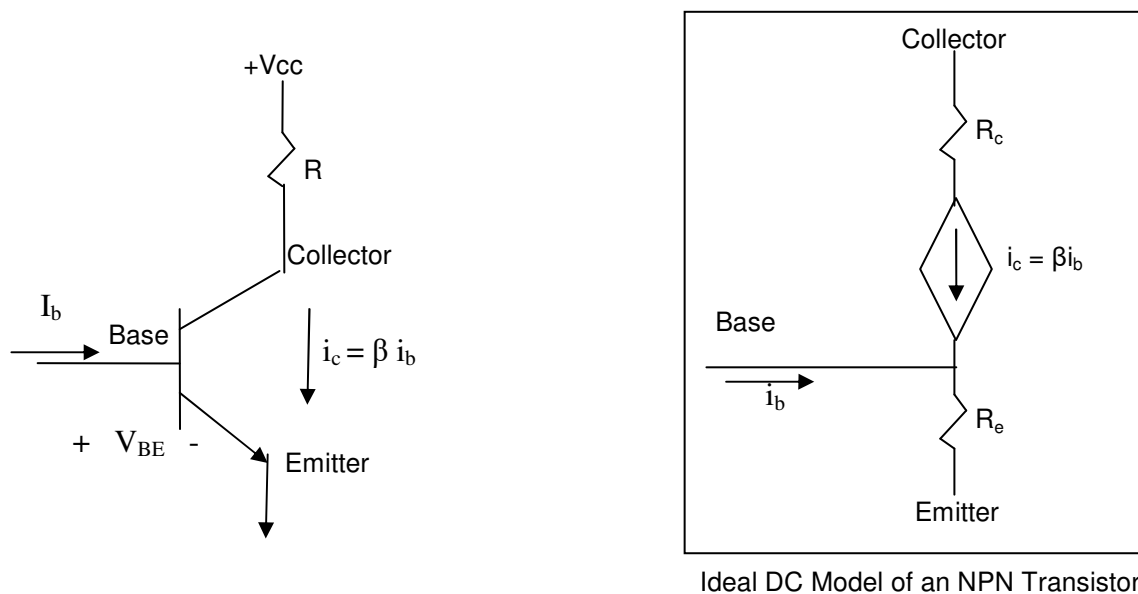
Transistors are implemented on a range of material with variety of polarities and configurations. In this introductory section on transistors, NPN transistor will be used to describe transistor fundamentals. NPN transistors are one of the earliest types of transistors. NPN Transistor was developed based on Silicon physical Characteristics with specialized processes to leverage and direct its characteristics.

Silicon, a semiconductor material, is turned into transistors through processes with most important one being doping where impurities are added. The doping results in material that either has extra electrons (which are called N-type for the extra negative charges) or add "holes" to the silicon's crystal structure (which is called P-type because it results in more positive charges).

The following diagram shows the NPN transistor circuit symbol and physical construction:

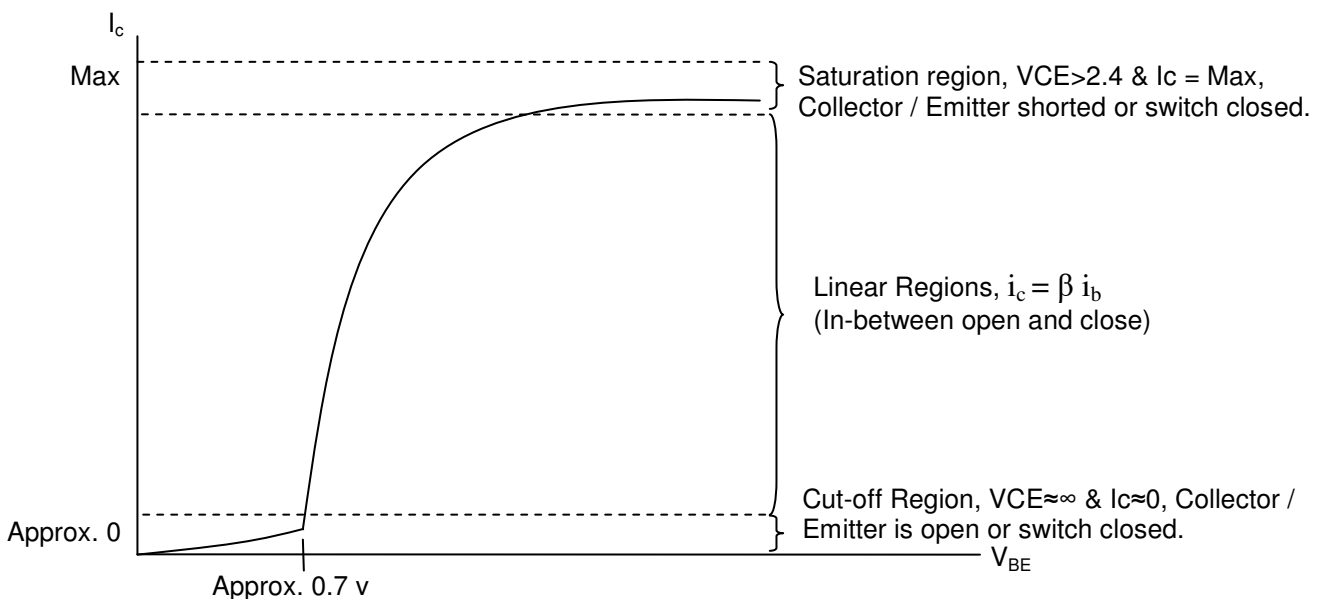


The following diagram utilizes a controlled current source and resistors to model the behavior of a transistor as an electrical circuit element. The model is referred to as the Ideal DC model for the Transistor.



As it is shown by the above transistor Ideal DC model, the current into the base (i_b) is amplified by a factor of β and flows from collector to emitter ($i_c = \beta i_b$). Typical β is around 300. This means in a typical case, 1 mA base current produces 300 mA Collector current.

The following chart shows the relationship between the V_{BE} and the current through the Collector Emitter Junction (I_c):



As shown above, Transistors operate in three distinct regions:

- Saturation Region
- Linear Region
- Cut-Off Region

Analog electronics use transistors in Linear mode which operate as an amplifier. In this type of applications, designer challenge is to design a circuit that remains in linear region and avoids the cut off and saturation region.

All digital and computer systems operate transistors in the cut off and saturation region with special focus on minimizing the amount of time spent in the linear regions. Since digital system utilize the transistor as a switch by applying $V_{BE} \approx 0$ to put the transistor in cut off region or open Collector / Emitter Connection (Switch open). For NPN transistor V_{BE} is increase as quickly as possible to above 3 volts to put the transistor in the saturation mode or short Collector / Emitter connection (Switch closed).

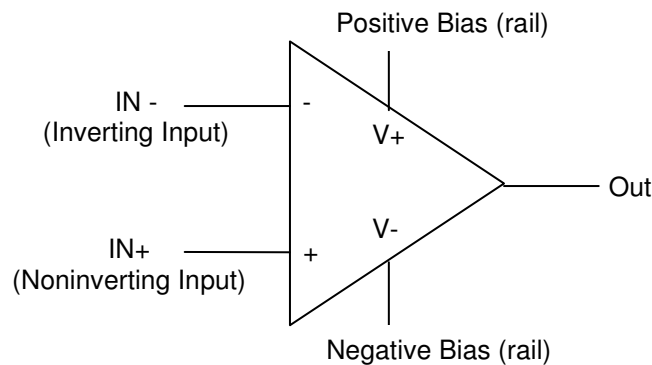
5.2. Operational Amplifier (Op Amp)

The Operational Amplifier (Op Amp) was initially used to implement mathematical operations such as summing and integration in analog computers. This initial application is where the Op Amp gets its name. Op Amp is a versatile component with wide variety of applications. Refer to LM 324 specification for an example of a general purpose low frequency low cost Op Amp.

The most common uses of Op Amp in the industry are:

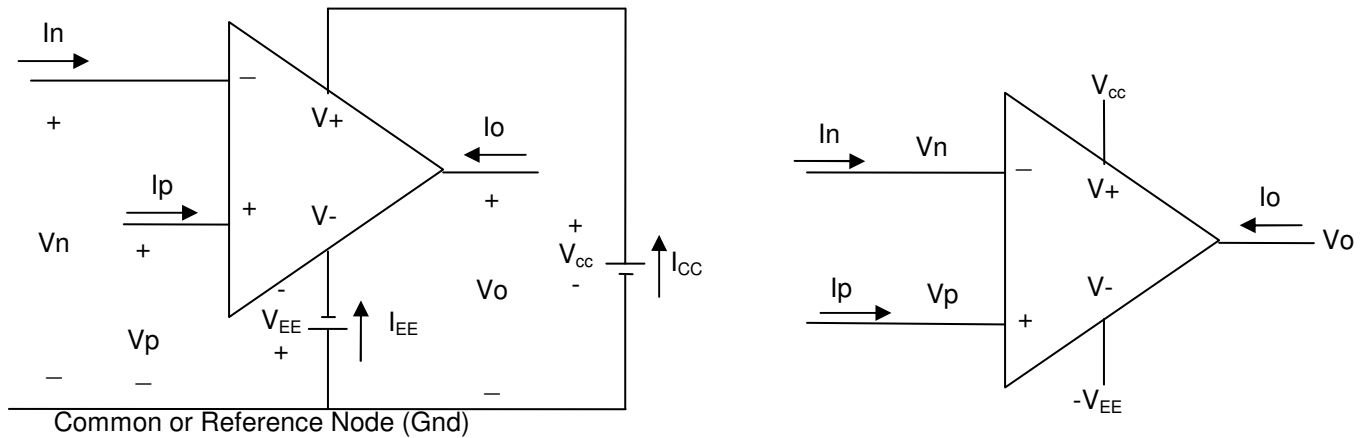
- 1) Signal Inversion
- 2) Signals Summation
- 3) Signals Subtraction
- 4) Signal Amplification
- 5) Signal Filtration

This section covers the application and external behavior of Op Amp in circuits but does not discuss the internal design of the Op Amps. In this text, the Op Amp is typically represented by the following symbol:



As shown above, the Op Amp has two inputs (positive & negative), either one or two bias supplies (also called rails since they provide the upper and lower output limits) and one output. When one bias supply is used such as in LM 324, the other bias is set to ground ($V=0$).

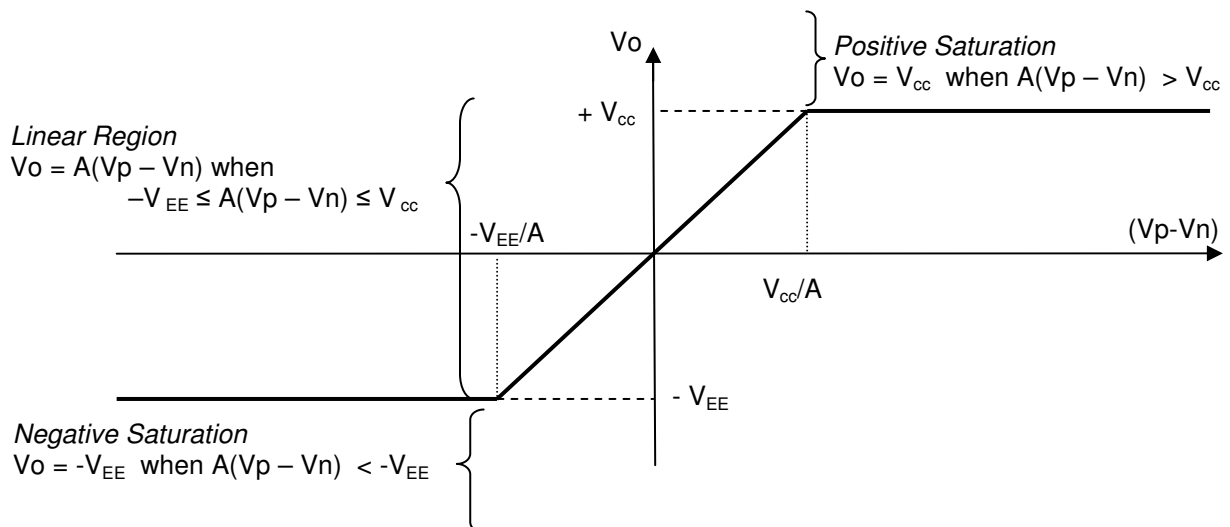
Typically, the simplified schematic which is use in the rest of the section, the supply connection to ground is not shown. This will simplify the schematic therefore making it easier to analyze the circuit. The following diagram shows a fully drawn schematic to the left and a simplified version to the right.



Open Loop Gain (A) is the value that input voltage difference ($V_p - V_n$) is multiplied by to get the output. A is called Open Loop Gain since it is the gain when input and output of Op Amp are not connected (open loop) and therefore there is no feedback. The value of A is typically well above the 10,000. In general, output and input are related based on the following equation as long as the output is within the supply range ($-V_{EE} \leq V_{cc} \leq V_{cc}$) which is also called the linear region:

$$V_o = A (V_p - V_n) \quad \text{if } -V_{EE} \leq V_o \leq V_{cc}$$

In open loop configuration, the relationship between the input voltages (V_p & V_n) and output voltage (V_o) depends on the Op Amp modes and relationship of $A(V_p - V_n)$ with supply voltage V_{cc} and $-V_{EE}$.



Open Loop Gain, A , for LM 324 is 100,000 at low frequencies and assuming $V_{cc} = 30$ v supply is used to bias the Op Amp then $(V_p - V_n)$ must be less than 0.3 mV in order for the LM 324 Op Amp to operate in the linear region as shown below:

$$\text{Linear Region requires that } A(V_p - V_n) < V_{cc} \rightarrow 100000(V_p - V_n) < 30 \rightarrow (V_p - V_n) < 0.3 \text{ mV}$$

This limitation is a major constraint since the smallest difference between V_p and V_n results in the Op Amp to the saturated and no longer operating as an amplifier. In order to extend the input range in the linear region, the designer needs to design the Op Amp with negative feedback which is discussed in latter part of this section.

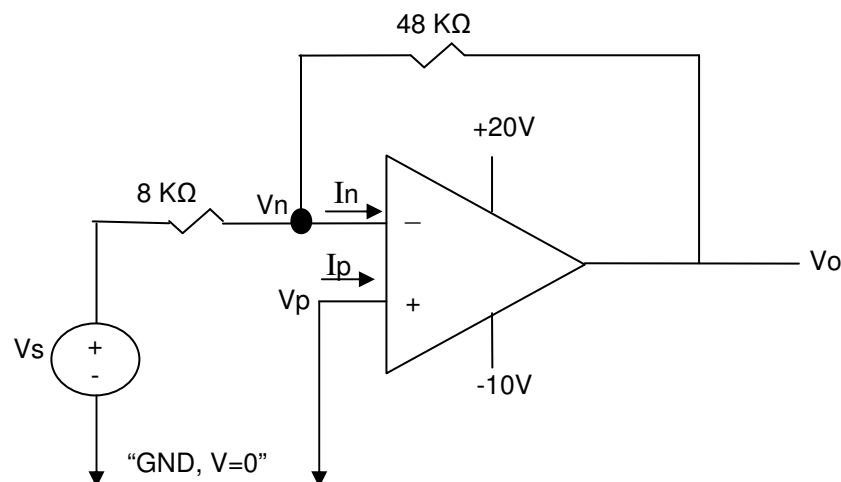
In order to analyze a circuit with an Op Amp, an equivalent Op Amp circuit or a model is required. The first model is the simplest one and is called the Ideal Op Amp Model. Although simple, the Ideal Op Amp Model is an effect tool in analyzing circuits with Op Amp. The Ideal Op Amp Model has the following three characteristics:

- 1) Open Loop Gain is infinite ($A = \infty$)
- 2) Voltage Constraint – Input Voltages are the same ($V_p - V_n = 0$ or $V_p = V_n$)
- 3) Current Constraint – Input is virtual open ($I_p = I_n = 0$)

Note: Characteristics 2 and 3 are based on the assumption that the input resistance (between input p and n) is infinite and no current flows through it.

Ideal Op Amp Model in conjunction with Kirchhoff Current Law (KCL) is the preferred approach to analyzing circuits with Op Amp in order to approximate the circuit's behavior.

- ❖ Example – Application of Ideal Op Amp Model to circuit analysis
For the following circuit:



Find V_o for the values of $V_s = 1.2, 6.0, -1.5$ and -5 .

Solution

KCL for Node $V_n \rightarrow (V_n - V_s)/8000 + (V_n - V_o)/48000 + I_n = 0$

From the circuit $\rightarrow V_p = 0$

Ideal Op Amp Characteristic $\rightarrow A = \infty, I_n = I_p = 0$ & $V_p = V_n$

Combine the above set of equations $\rightarrow -V_s/8000 - V_o/48000 = 0 \rightarrow V_o = -6V_s$

$V_s = 1.2 \rightarrow V_o = -7.2$

$V_s = 6.0 \rightarrow V_o = -36$ (this is less than $-V_{EE} = -10V$ so Saturates) $\rightarrow V_o = -10$

$V_s = -1.5 \rightarrow V_o = 9.0$

$V_s = -5 \rightarrow V_o = 30$ (this is more than $V_{cc} = 20V$ so Saturates) $\rightarrow V_o = 20$

Another way is to calculate the range in which the Op Amp saturates as shown below:

$$-10 \leq V_o \leq 20 \rightarrow -10 \leq -6V_s \leq 20 \rightarrow -3.3 \leq V_s \leq 1.7$$

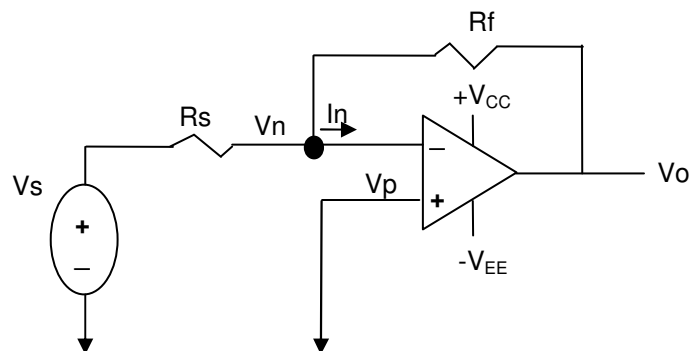
5.3. Op Amp Applications

Op Amps are typically used to invert, amplify, add, subtract and compare signals in a circuit. This section introduces the basic design and analysis approach for this type of applications. Throughout this section, the Ideal Op Amp Model is used to analyze these circuits.

❖ Inverting-Amplifier Circuit

Inverting-Amplifier negates and scales (amplifies or attenuates) source voltage (V_s) and outputs it as Output Voltage (V_o).

➤ Circuit Diagram



➤ Circuit Analysis

$$\text{KCL Node } V_n \rightarrow (V_n - V_s)/R_s + (V_n - V_o)/R_f + I_n = 0$$

$$\text{Ideal Op Amp Characteristics} \rightarrow A = \infty, V_p = V_n, I_p = I_n = 0$$

$$\text{From the circuit } V_p = 0$$

$$\text{Combine the above equation set} \rightarrow -V_s/R_s - V_o/R_f = 0 \rightarrow V_s/R_s = -V_o/R_f$$

$$\mathbf{V_o = - (R_f/R_s) * V_s} \text{ where:}$$

“-“ indicates an inverting amplifier

“ R_f/R_s ” is the scaling factor or gain

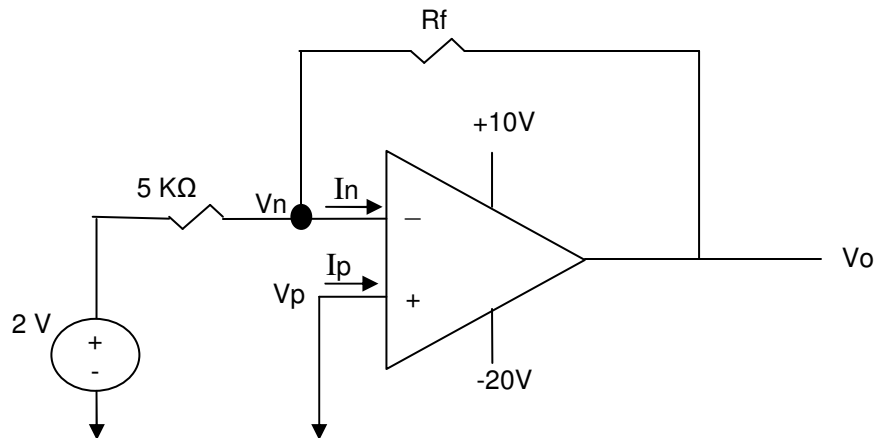
If $R_f/R_s > 1$ then signal is amplifier

If $R_f/R_s < 1$ then signal is attenuated

The above relationship is valid only in the linear region where $-V_{EE} \leq V_o \leq V_{CC}$ or $-V_{CC} \leq -(R_f/R_s) * V_s \leq V_{CC}$. As discussed earlier, outside of linear regions, V_o will be limited by either the V_{CC} or $-V_{EE}$ rail..

➤ Example - Inverting amplifier

For the following circuit:



Find the range of R_f values that keeps the amplifier in the linear region.

Solutions

KCL Node $V_n \rightarrow (V_n - 2)/5000 + (V_n - V_o)/R_f + I_n = 0$

Ideal Op Amp Characteristics $\rightarrow A = \infty, V_p = V_n, I_p = I_n = 0$

From the circuit $\rightarrow V_p = 0$

Combining the above set of equations $\rightarrow -2/5000 + (-V_o/R_f) = 0 \rightarrow V_o = -(2/5000)R_f$

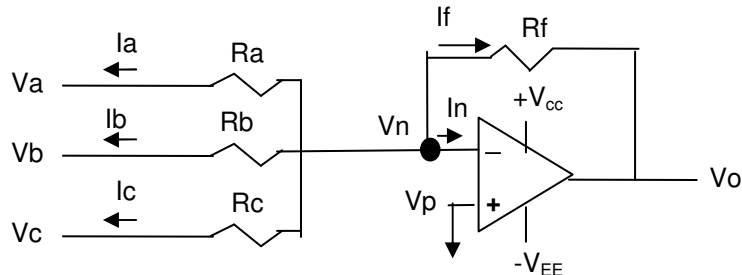
For the Op Amp to be in Linear Region $\rightarrow -20 \leq V_o \leq 10$

Substitute V_o equivalent in term of R_f $V_o \rightarrow -20 \leq -(2/5000)R_f \leq 10 \rightarrow 50000 \geq R_f \geq -25000$
 since R cannot be less than 0 lower limit is 0 \rightarrow

$R_f \leq 50 \text{ K}\Omega$ for Op Amp to operate in linear region

- ❖ **Summing-Amplifier Circuit**
Sums a number of voltage sources (V_a , V_b , ...) and output the scaled (amplified or attenuated) result as the Output Voltage (V_o).

➤ **Circuit Diagram**



➤ **Circuit Analysis**

KCL at the Node $n \rightarrow (V_n - V_a)/R_a + (V_n - V_b)/R_b + (V_n - V_c)/R_c + I_n + (V_n - V_o)/R_f = 0$

Ideal Op Amp $\rightarrow I_n = I_p = 0$ & $V_p = V_n$

From the circuit $\rightarrow V_p = 0$

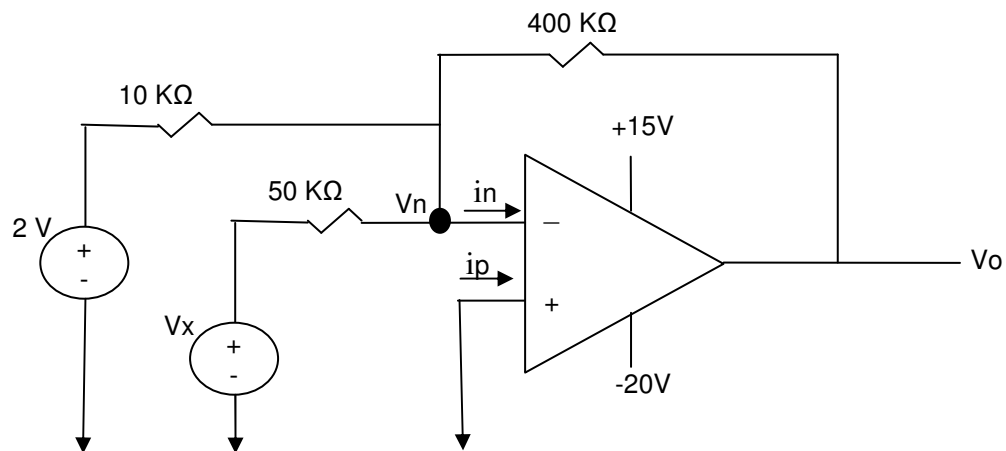
Combining the above set of equations $\rightarrow V_a/R_a + V_b/R_b + V_c/R_c + 0 + V_o/R_f = 0 \rightarrow$

$$\mathbf{V_o = -R_f * (V_a/R_a + V_b/R_b + V_c/R_c)}$$

The above relationship is valid only in the linear region where $-V_{EE} < V_o < V_{cc}$

➤ **Example – Summing-Amplifier**

For the following circuit:



Find the largest value of V_x while Op Amp is in linear region.

Solution

$$V_o = -R_f * (V_a/R_a + V_b/R_b) = -400(2/10 + V_x/50) = -80 - 8V_x$$

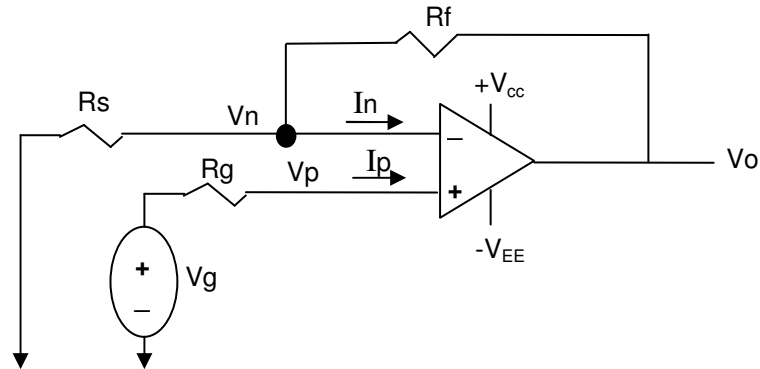
$$\text{Linear region} \rightarrow -20 \leq V_o \leq 15 \rightarrow -20 \leq -80 - 8V_x \leq 15$$

$$\text{Simplify} \rightarrow 60 \leq -8V_x \leq 95 \rightarrow -7.5 \geq V_x \geq -11.9$$

The largest value of V_b that keeps the Op Amp in Linear Region (not saturated) is -7.5 V

- ❖ **Non-inverting-Amplifier Circuit**
Non-inverting-Amplifier scales (amplified or attenuated) the source voltage (V_g) and outputs it as Output Voltage (V_o).

➤ **Circuit Diagram**



➤ **Circuit Analysis**

KCL at the Node $n \rightarrow V_n/R_s + (V_n - V_o)/R_f + I_n = 0$

Ideal Op Amp $\rightarrow I_n = I_p = 0$ & $V_p = V_n$

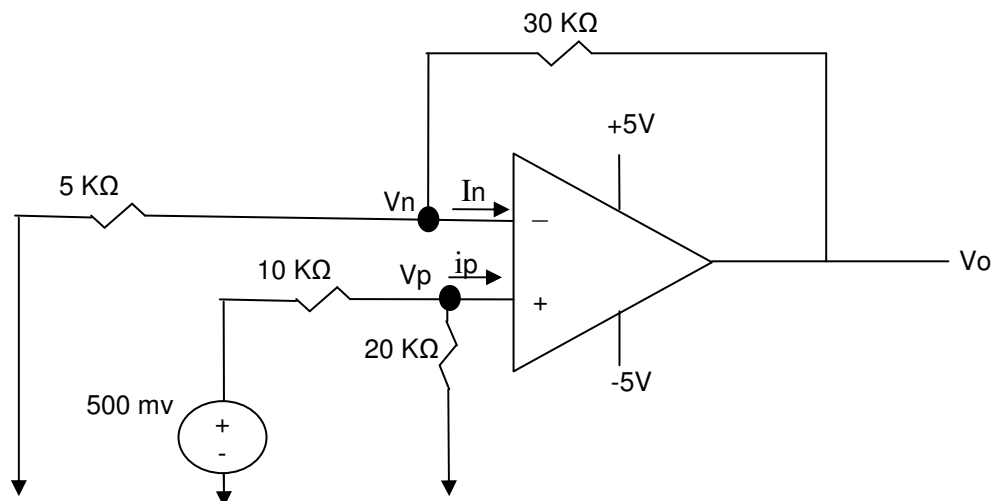
From the circuit $\rightarrow V_p = V_g$ since $I_p = 0$ then there is no voltage drop across R_g

Combine above equation set $\rightarrow V_g/R_s + (V_g - V_o)/R_f = 0 \rightarrow V_o = \{(R_s + R_f)/R_s\} * V_g$

$\{(R_s + R_f)/R_s\}$ is the scaling factor and the above relationship is valid only in the linear region where $-V_{EE} \leq V_o \leq V_{CC}$ or $-V_{EE} \leq \{(R_s + R_f)/R_s\} * V_g \leq V_{CC}$.

➤ **Example – Non-inverting-Amplifier**

For the following circuit:



Find the Output Voltage, V_o .

Solutions

Ideal Op Amp $\rightarrow I_n = I_p = 0$ & $V_p = V_n$

KCL at the Node $n \rightarrow V_n/5 + (V_n - V_o)/30 + I_n = 0$

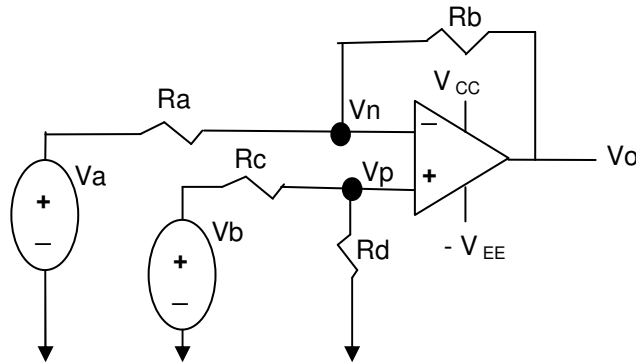
KCL at the Node $p \rightarrow (V_p - 0.5)/10 + V_p/20 + I_p = 0 \rightarrow V_p = 1/3 \text{ V}$

Combine the above set of equation $\rightarrow 1/15 + (1/3 - V_o)/30 = 0 \rightarrow V_o = 2.33 \text{ V}.$

❖ The Difference Amplifier Circuit

The output voltage of difference amplifier is proportional to the difference between the two input voltages.

➤ Circuit Diagram



➤ Circuit Analysis

KCL at the Node n $\rightarrow (V_n - V_a)/R_a + (V_n - V_o)/R_b + I_n = 0$

KCL at the Node p $\rightarrow (V_p - V_b)/R_c + V_p/R_d + I_p = 0$

Ideal Op Amp $\rightarrow I_n = I_p = 0$ & $V_p = V_n$

Combine the above set of equations \rightarrow

$$(V_p - V_a)/R_a + (V_p - V_o)/R_b = 0$$

$$V_p = \{R_d/(R_c + R_d)\}V_b$$

$$\text{Plug value of } V_p \text{ in the first equation} \rightarrow V_o = \frac{R_d(R_a + R_b)}{R_a(R_c + R_d)}V_b - \frac{R_b}{R_a}V_a$$

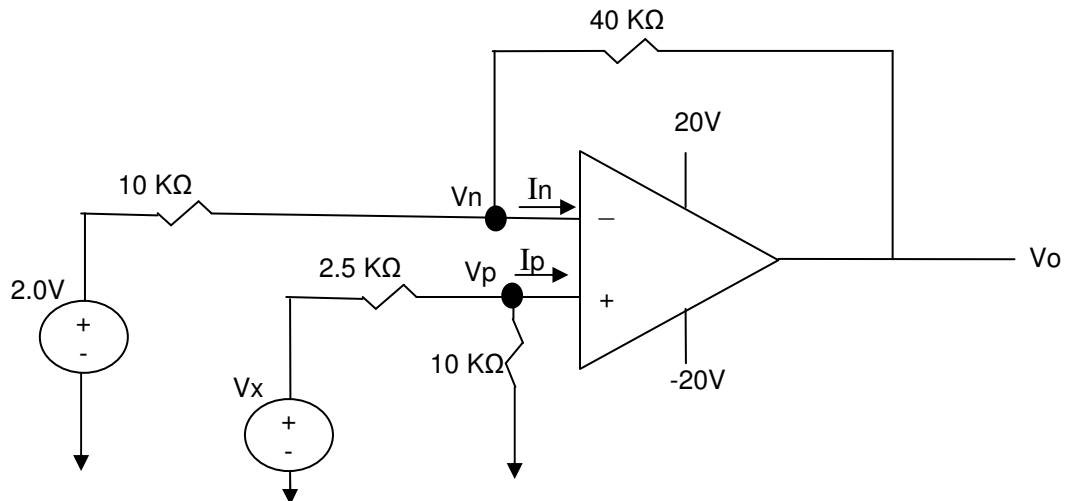
A useful simplified special case is where $k = R_b/R_a = R_d/R_c$ simplifies the equation to:

$$V_o = k(V_b - V_a)$$

The above relationship is valid only in the linear region where $-V_{EE} < V_o < V_{CC}$

➤ Example - Difference Amplifier

For the following circuit:



Find the range of V_x values that will result in Linear operation (Not Saturated) of OpAmp.

$$\text{KCL at } V_n \rightarrow (V_n - 2)/10 + (V_n - V_o)/40 = 0 \rightarrow 5V_n - 8 = V_o$$

$$\text{KCL at } V_p \rightarrow (V_p - V_x)/2.5 + V_p/10 = 0 \rightarrow 5V_p - 4V_x = 0 \rightarrow V_p = 4V_x/5$$

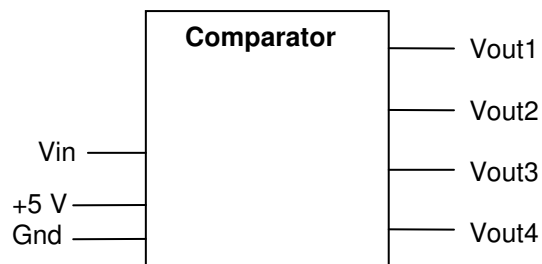
Ideal Op Amp Characteristics $I_n = I_p = 0$; $V_n = V_p$

$$\text{Combine the above equations} \rightarrow V_o = 5(4V_x/5) - 8 \rightarrow V_o = 4V_x - 8$$

$$\text{Linear region} \rightarrow -20 \leq V_o \leq 20 \rightarrow -20 \leq 4V_x - 8 \leq 20 \rightarrow -3 \leq V_x \leq 7$$

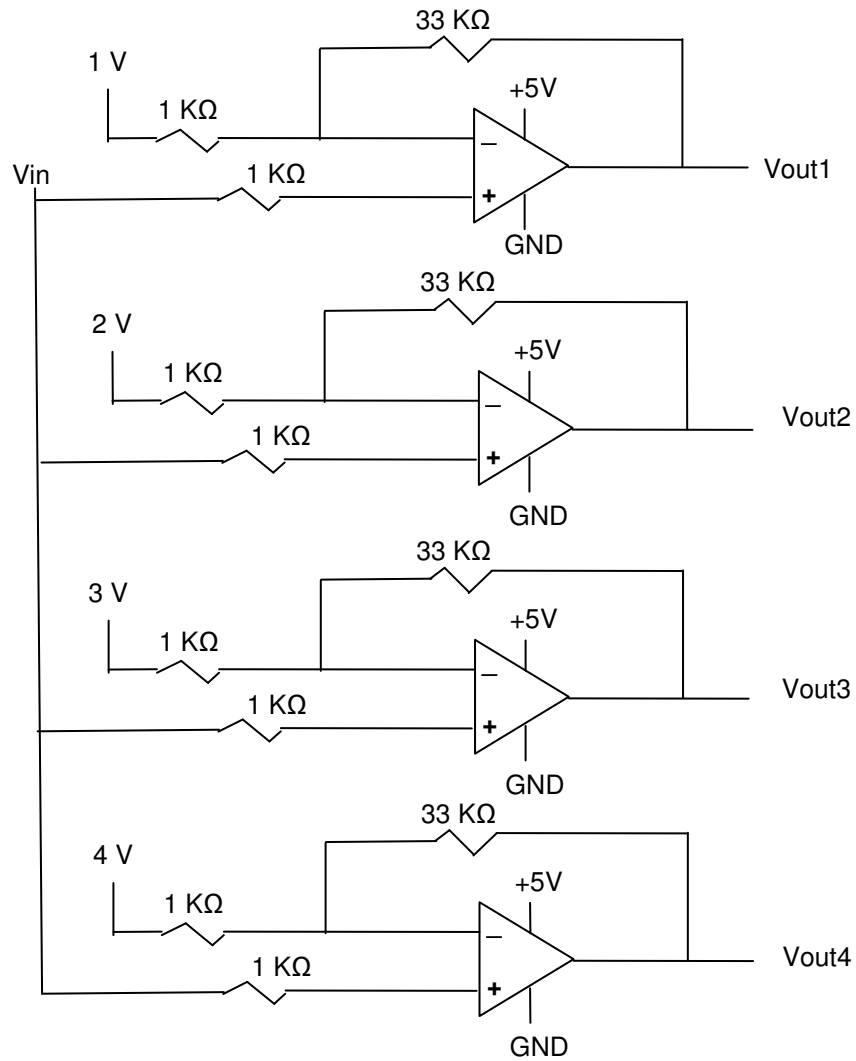
❖ Example – Build a comparator using ideals Op Amps such that it meets the following conditions (0.2 volts of tolerance is allowed)::

- $V_{out1} > 3.5$ Volts when $V_{in} > 1$ v , otherwise $V_{out1} < 0.4$
- $V_{out2} > 3.5$ Volts when $V_{in} > 2$ v , otherwise $V_{out2} < 0.4$
- $V_{out3} > 3.5$ Volts when $V_{in} > 3$ v , otherwise $V_{out3} < 0.4$
- $V_{out4} > 3.5$ Volts when $V_{in} > 4$ v , otherwise $V_{out4} < 0.4$



Solution

This problem can be solved using 4 Non-inverting Amplifiers as shown below:

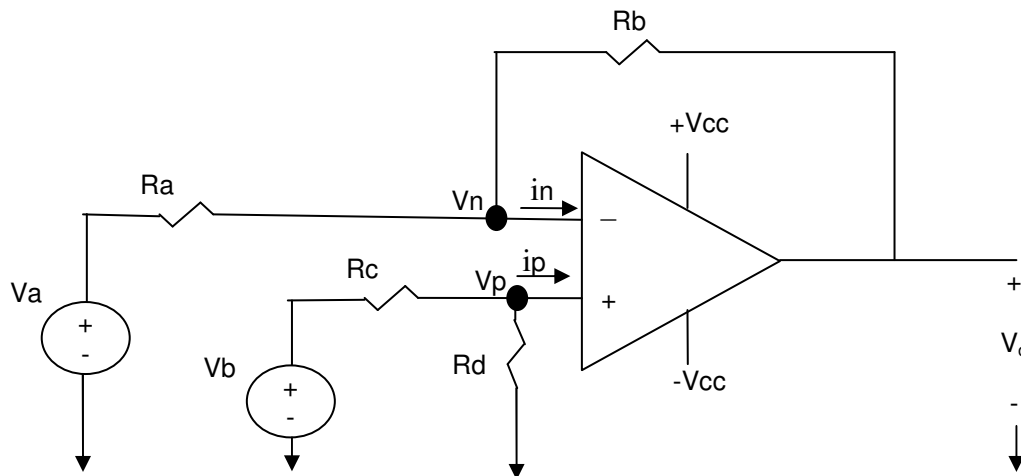


5.4. Common Mode (cm) vs. Differential Mode (dm)

Op Amp's two input present two types of voltages. One is the difference between the positive and negative input voltage which is called the Differential Mode voltage and the other is the difference between the input and the ground or common) which is called the Common Mode voltage.

In most application, the signal is carried by the Differential Mode voltage and the Common Mode voltage is the result of unwanted energy (noise) from other electrical sources in the environment. So it is important to minimize the impact of Common Mode voltage (noise) and maximize the impact of Differential Mode voltage.

The first step is to redraw Op Amp circuits so that the input is represented in terms of Common Mode and Differential Mode voltage. Starting with the Difference-Amplifier circuit shown below:



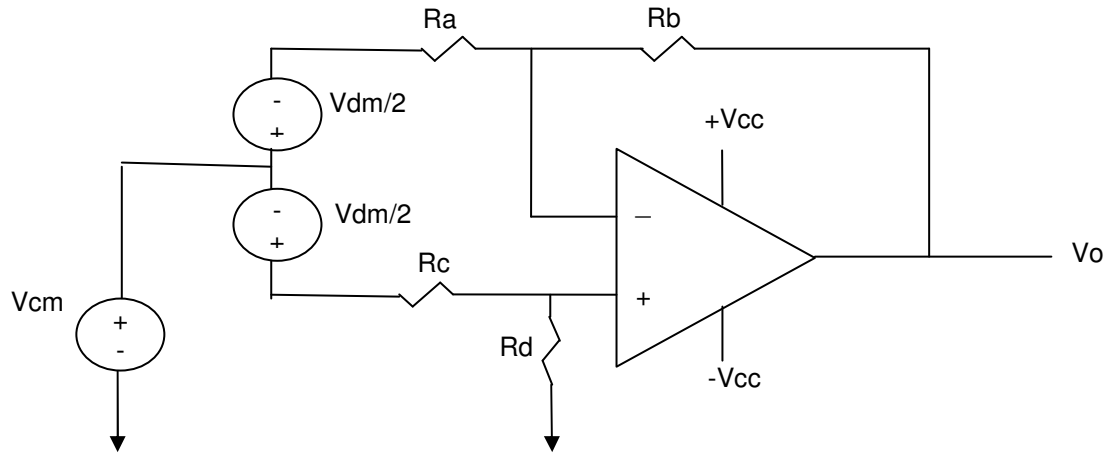
Applying the definition of Differential Model and Common Mode voltages results in the following two equations:

- 1) $V_{dm} = V_b - V_a$ (Differential Mode voltage - Signal)
- 2) $V_{cm} = (V_b + V_a)/2$ (Common mode voltage - Noise)

Rewriting the equation so that the value of voltage sources are written in terms of V_{cm} and V_{dm} results in the following two equations::

- 1) $V_a = V_{cm} - (1/2)V_{dm}$
- 2) $V_b = V_{cm} + (1/2)V_{dm}$

Using the above two equation, the initial Difference-Amplifier can be redrawn as:



From earlier section, it is known that for a Difference amplifier, the following relationship is correct:

$$V_o = \frac{R_d(R_a + R_b)}{R_a(R_c + R_d)} V_b - \frac{R_b}{R_a} V_a$$

By substituting for V_a and V_b with their equivalent in terms of V_{cm} and V_{dm} , the output voltage equation may be rewritten as:

$$V_o = \left[\frac{R_a R_d - R_b R_c}{R_a(R_c + R_d)} \right] V_{cm} + \left[\frac{R_d(R_a + R_b) + R_b(R_c + R_d)}{2R_a(R_c + R_d)} \right] V_{dm}$$

$$V_o = A_{cm} V_{cm} + A_{dm} V_{dm}$$

Where:

$$A_{cm} = \left[\frac{R_a R_d - R_b R_c}{R_a(R_c + R_d)} \right] \text{ Common Mode (Noise) Gain}$$

$$A_{dm} = \left[\frac{R_d(R_a + R_b) + R_b(R_c + R_d)}{2R_a(R_c + R_d)} \right] \text{ Differential Mode (Signal) Gain}$$

An Ideal Difference Amplifier has a Common Mode Gain of zero ($A_{cm} = 0$). This is an Ideal Difference Amplifier since it eliminates the impact of Common Mode voltage (Noise) from the output.

Ideal Difference Amplifier definition ($A_{cm} = 0$) $\rightarrow R_a R_d - R_b R_c = 0 \rightarrow R_c = R_a$ & $R_d = R_b$

which means the output equation simplifies to the following equation for an Ideal Difference Amplifier:

$$V_o = \frac{R_b}{R_a} V_{dm}$$

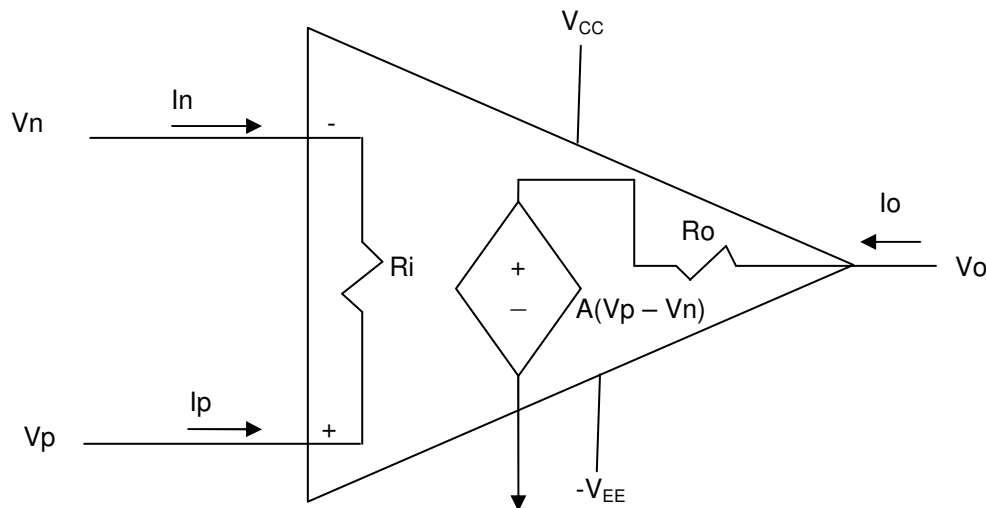
The goodness of a Non-Ideal Difference Amplifier is measured based on the value of Common Mode Rejection Ratio ($CMRR = |A_{dm}/A_{cm}|$). The larger CMRR, the closer the ideal Difference-Amplifier. In other word, $CMRR = \infty$ represent an ideal Differential Amplifier which Occurs when either V_{cm} is infinite or V_{dm} is zero..

5.5. Op Amp DC Model

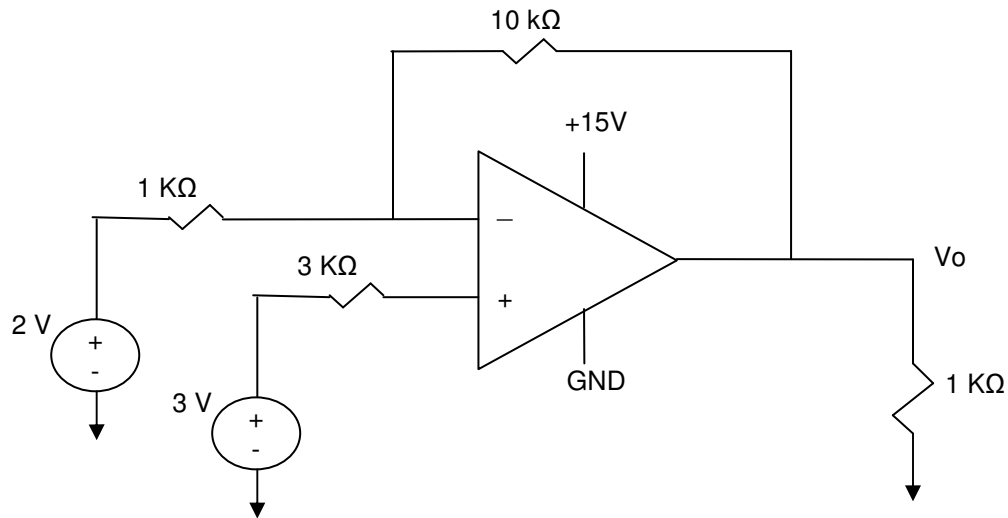
Although the Ideal Op Amp model provides a simple model for analyzing OP Amp circuits, some application required better approximation. In this section, a more realistic model for the Op Amp is introduced which is referred to as the DC Model. Here are three differences between DC and Ideal Model of OP Amp:

- 1) Ideal Model assumed an infinite input resistance, DC model uses the input resistance which can be found in the Op Amp Specifications (for example: R_i for LM 324 is $2\text{ M}\Omega$)
- 2) Ideal Model assumed an infinite open loop gain, DC model uses the value from the Op Amp Specifications (for example: A for LM 324 is 100,000)
- 3) Ideal Model assumed an output resistance of zero, DC model uses the value from the Op Amp Specifications (for example: R_o for LM 324 is $75\ \Omega$)

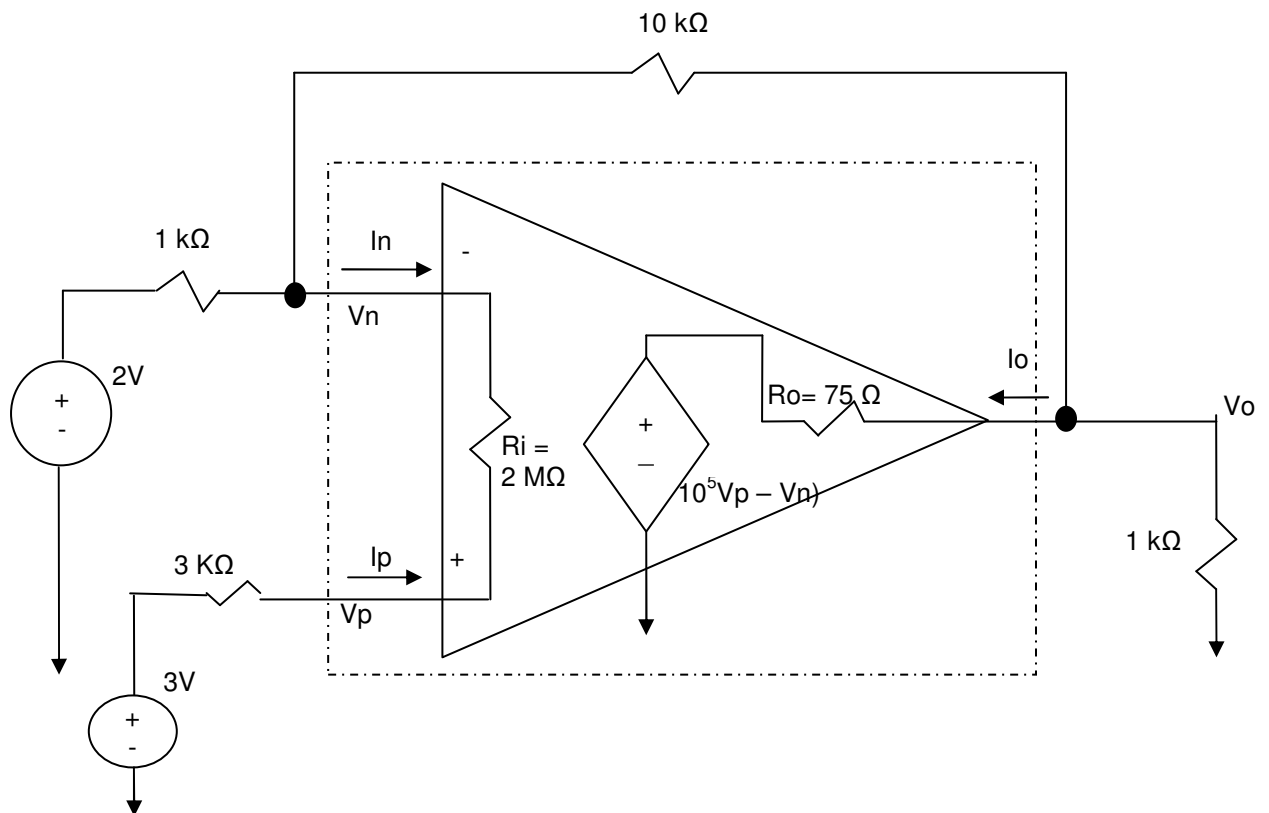
It is understood that the ideal Op Amp Model assumptions such as $V_p = V_n$ and $I_p = I_n = 0$ are not valid when using the DC model. The following circuit diagram shows the equivalent circuit for Op Amp when using a DC Model:



- Example – OpAmp DC Model
Use OpAmp DC Model ($R_o=75\Omega$ and $R_i = 2\text{ M}\Omega$) to find V_o for the following circuit:

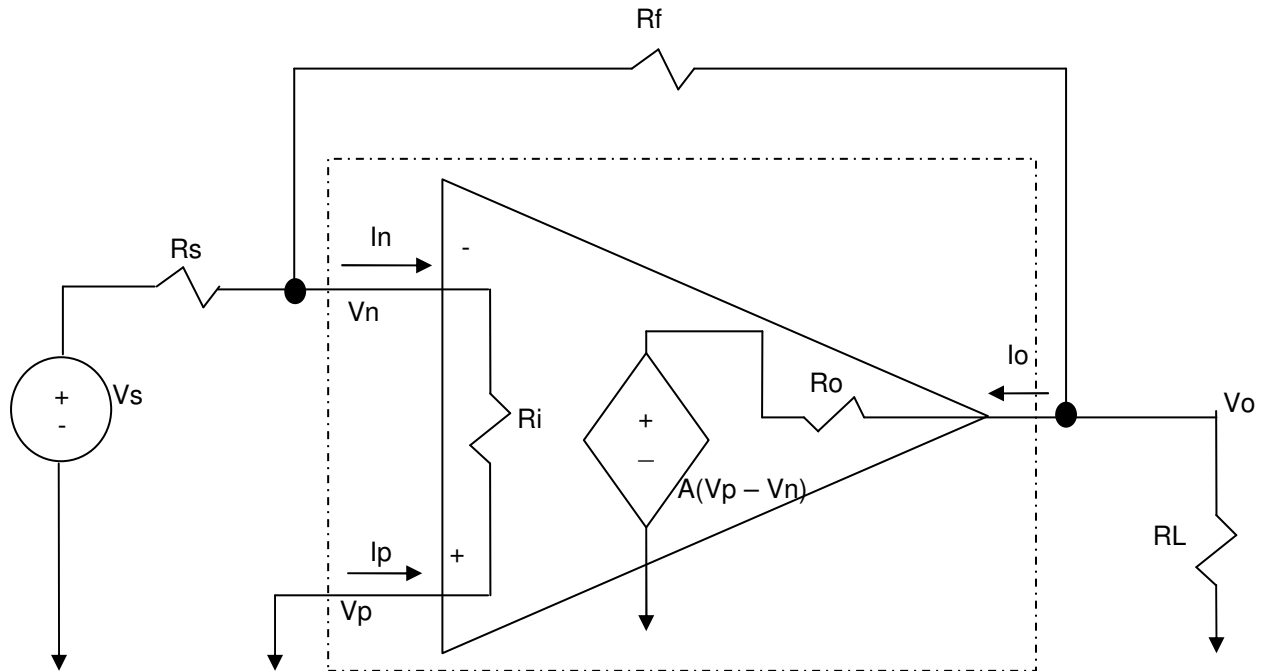


Solutions



$$\begin{aligned} \text{KCL @ } V_n &\rightarrow (V_n - 2) + (V_n - V_o)/10 + (V_n - 3)/2003 = 0 \\ \text{KCL @ } V_o &\rightarrow (V_o - V_n)/10 + V_o - 105(V_p - V_n)/0.075 + V_o = 0 \\ \text{Where } V_p &= 3 + 3(V_n - 3)/2003 \end{aligned}$$

- Example – OpAmp DC Model
Find V_o for the following circuit using the DC model of Op Amp:



Form the circuit $\rightarrow V_p = 0$

$$\text{KCL at } V_n \rightarrow (V_n - V_s)/R_s + V_n/R_i + (V_n - V_o)/R_f = 0$$

$$\text{KCL at } V_o \rightarrow (V_o - V_n)/R_f + (V_o - A(V_p - V_n))/R_o + V_o/R_L = 0$$

Simplify the equations

$$(1/R_s + 1/R_i + 1/R_f)V_n - V_o/R_f - V_s/R_s = 0 \rightarrow V_o = R_f \{ (1/R_s + 1/R_i + 1/R_f)V_n - V_s/R_s \}$$

$$(1/R_f + 1/R_o + 1/R_L)V_o - (1/R_f - A/R_o)V_n = 0 \rightarrow V_n = (1/R_f + 1/R_o + 1/R_L)/(1/R_f - A/R_o)$$

Plug V_n into first equation

$$V_o = R_f \{ (1/R_s + 1/R_i + 1/R_f)(1/R_f + 1/R_o + 1/R_L)/(1/R_f - A/R_o) - V_s/R_s \}$$

5.6. Summary

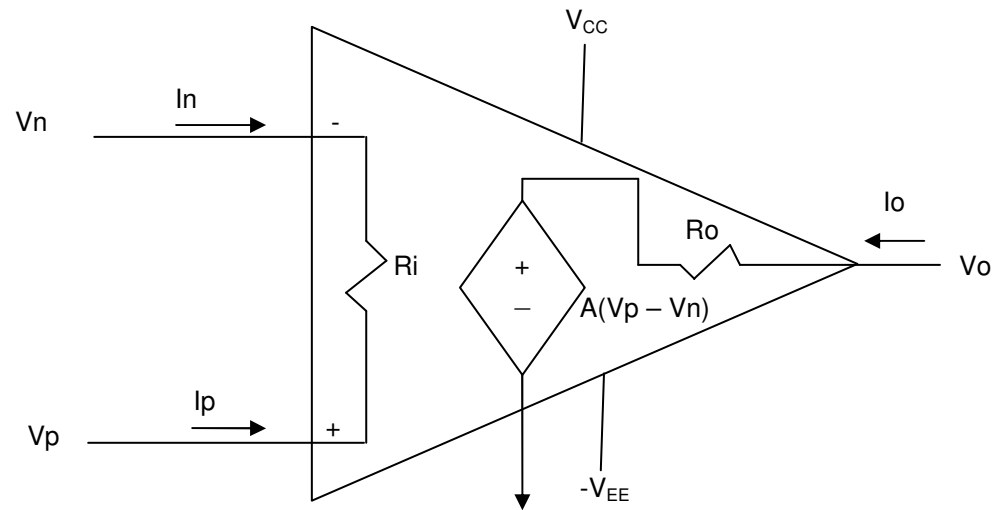
❖ Ideal Op Amp Model

$$A = \infty$$

$$V_p = V_n$$

$$I_p = I_n = 0$$

❖ DC Op Amp Model



❖ Common & Differential Mode

$$V_o = \left[\frac{R_a R_d - R_b R_c}{R_a (R_c + R_d)} \right] V_{cm} + \left[\frac{R_d (R_a + R_b) + R_b (R_c + R_d)}{2 R_a (R_c + R_d)} \right] V_{dm}$$

$$V_o = A_{cm} V_{cm} + A_{dm} V_{dm}$$

$$CMRR = \left| \frac{A_{dm}}{A_{cm}} \right| \quad \text{Common Mode Rejection Ratio}$$

5.7. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 5.

5.8. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 6. Capacitors and Inductors

Key Concepts and Overview

- ❖ Passive Elements
- ❖ Inductor, L
- ❖ Series and Parallel Inductors
- ❖ Capacitor, C
- ❖ Series and Parallel Capacitors
- ❖ Additional Resources

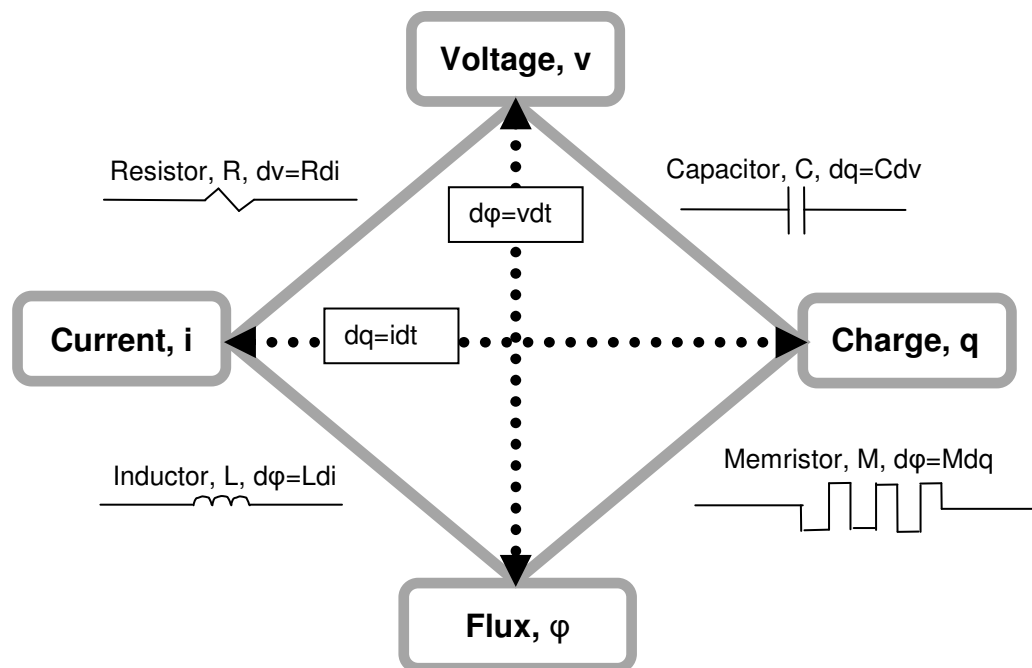
6.1. Passive Elements

In the study of electricity, electrical components are classified either as passive elements or active elements. Passive elements are the elements that consume energy. In other words, they are not capable of power gain. On the contrary, active elements are the elements that can produce energy or are capable of power gain.

So far four active elements have been introduced which are current source, voltage source, transistor and OpAmp. And the only passive element introduced is resistor which relates voltage and current. In addition to resistor, inductor and capacitor are also passive elements that are well understood and will be introduced in detail later in this section. The fourth element is called a Memristor and in 1971, Leon Chua theorized its existence but it was not until 2008 that a working memristance was developed by scientists at Hewlett Packard.

The four fundamental circuit variables are current, i ; voltage, V , charge, Q ; and magnetic flux, Φ . The first three circuit variables have already been introduced in earlier chapters. Magnetic flux, represented by the Greek letter Φ (phi), is a measure of quantity of magnetism. Magnetic flux is measured in units of weber (volt-seconds) per square meter, or tesla. Magnetic field surrounds electric current so if charge is traveling through a wire (current), it will result in a magnetic field that would surround the wire.

Each of the four basic passive components relates two of the four circuit variables as shown below:



You may have noticed that both capital and lower case letters are used to represent circuit variables such as current and voltage. In electrical engineering, capital letters are used to refer to circuit variables when they are constant with respect to time. Prior to this section most of the circuits analysis was done with constant circuit variables. In this section, circuit variables may not be constant with respect to time and therefore lower case letters will be used as it is done in the above overview diagram of relationship between electrical circuit elements and variables.

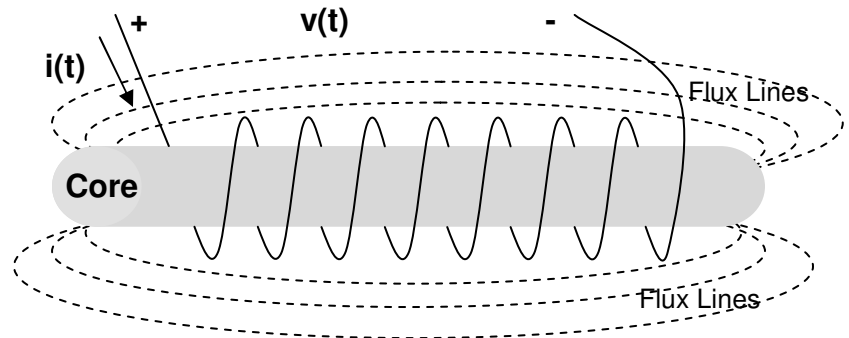
Remainder of this chapter introduces inductor, L , and capacitor, C .

6.2. Inductor, L

Inductor is traditionally built by winding a coil of wire around a supporting core which its material may or may not be magnetic. In some cases, the core is simply air (a nonconducting - dielectric). Inductors are manufactured from variety of material and are packaged based on the application requirement so it is difficult to know an inductor's characteristic without the manufacturer's specifications.

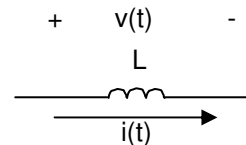
Below is a basic inductor construction for the purpose of demonstrating the underlying physical phenomenon and corresponding schematics:

Physical Construction →



Circuit Symbol →

Note that passive convention is used.



“henry” is the units used to measure inductance. The henry (symbol: H) is named after Joseph Henry, the 18th century American scientist who discovered electromagnetic induction (see flux lines) about the same time as Michael Faraday made the similar discovery in England. Faraday is not forgotten, Farad which is units of capacitance measurement is named after him.

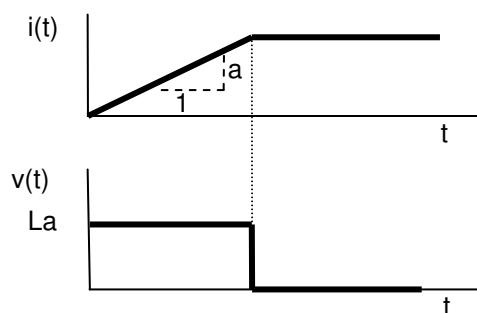
As electric current flows through a wire (charge moving), the charge movement causes the generation of electromagnetic field (flux) which induces voltage across the inductor terminals. The resulting voltage is derived by the following equation:

$$v(t) = L \frac{di(t)}{dt} \text{ When}$$

Voltage, $v(t)$, is in Volts
Current, $i(t)$, is in Amps
Inductance, L , is in henrys

As it can be seen from the above equation, voltage is only generated when the current changes. Here are a few observations:

- 1) If current is constant then the voltage across the inductor is Zero.
- 2) If the Current is changed from one level to another in time interval $dt \rightarrow 0$ then $v(t) \rightarrow \infty$.
- 3) If the current changes linearly then voltage remain constant as shown in the following diagram:



Application of introductory calculus is used to calculate inductor current in terms of voltage as shown below:

$$\text{Integrate } v(t) = L \frac{di(t)}{dt} \text{ from } t_0 \text{ to } t$$

$$i(t) = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0) \quad \text{Where } i(t_0) \text{ is value of inductor current at } t_0 \text{ also called initial condition}$$

The Power and Energy calculation in inductor are shown here

- Inductor Power Calculation, $p(t)$
Start with $p(t) = v(t) * i(t)$ and substitute equivalent of $v(t)$ in terms of $i(t)$

$$p(t) = Li(t) \frac{di(t)}{dt}$$

- Inductor Energy or Work Calculation, $w(t)$

$$p(t) = \frac{dw(t)}{dt} \quad \text{Power is change in energy(work)}$$

$$w(t) = \int p(t) dt = \int Li(t) \frac{di(t)}{dt} dt$$

$$w(t) = \frac{1}{2} Li^2(t)$$

where $w(t)$ is in Joules, L is in henrys and $i(t)$ is in Amperes

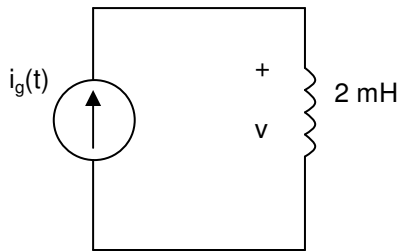
As it can be seen from the above power and energy calculation, inductor stores energy that may be delivered over time. This is different from resistor which did not have the ability to store electrical energy and would instantly react to changes in current or voltage.

❖ Example – Application of Inductor basic relationships

The current source in the following circuit generates a current, i_g :

$$i_g(t) = 0 \quad t < 0$$

$$i_g(t) = 12e^{-200t} - 12e^{-800t} \quad t \geq 0$$



Find:

- $v(t)$ immediately after $t=0$ which is referred to as $v(0^+)$
- $t>0$ when voltage $v(t)$ passes through zero.
- The expression for the power delivered to inductor
- Power delivered at $t= 25$ msec.
- Total energy stored at $t=25$ msec.

Solutions

Part a) Find $v(t)$ immediately after $t=0$ which is referred to as $v(0^+)$

$$v(t) = L \frac{di_g(t)}{dt} = 0.002 \frac{d(12e^{-200t} - 12e^{-800t})}{dt}$$

$$v(t) = 0.002(-2400e^{-200t} + 9600e^{-800t})$$

$$v(0^+) = 0.002(-2400e^0 + 9600e^0) = 14.4v$$

Part b) Find $t>0$ when voltage $v(t)$ passes through zero.

$$v(t) = 0.002(-2400e^{-200t} + 9600e^{-800t}) = 0$$

$$2400e^{-200t} = 9600e^{-800t}$$

$$1 = 4e^{-600t}$$

Take natural Log of both side

$$\ln(1/4) = \ln(e^{-600t})$$

$$-1.39 = -600t \ln e \quad \text{note : } \ln e = 1$$

$$t = 0.0023 \text{ sec} = 2.3 \text{ msec}$$

Part c) The expression for the power delivered to inductor

$$p(t) = v(t) * i(t)$$

$$p(t) = 0.002(-2400e^{-200t} + 9600e^{-800t})(12e^{-200t} - 12e^{-800t})$$

$$p(t) = -57.6e^{-400t} + 288.0e^{-1000t} - 230.4e^{-1600t}$$

Part d)) Power delivered to the inductor at $t= 25$ msec.

$$p(t) = -57.6e^{-400t} + 288.0e^{-1000t} - 230.4e^{-1600t}$$

$$p(t = 0.025) = -57.6e^{-400*(0.025)} + 288.0e^{-1000*(0.025)} - 230.4e^{-1600*(0.025)}$$

$$p(0.025) = -0.0026 \text{ W} = 2.6 \text{ mW}$$

Part e) Total energy stored in the inductor at t=25 msec.

$$w(t) = \frac{1}{2} Li(t)^2$$

$$w(t) = \frac{1}{2} * (0.002) * (12e^{-200t} - 12e^{-800t})^2$$

$$w(t = 0.025) = \frac{1}{2} * (0.002) * (12e^{-200*0.025} - 12e^{-800*0.025})^2$$

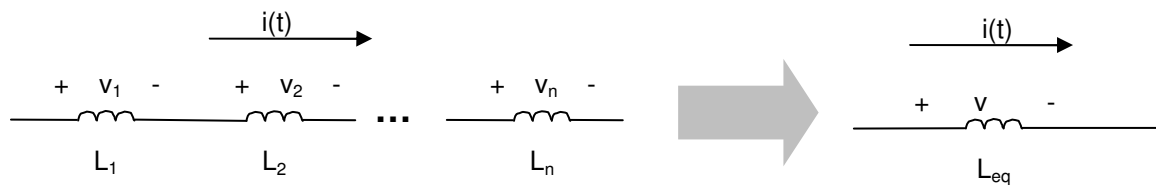
$$= 0.0000065 = 6.5 \mu J$$

6.3. Series and Parallel Inductors

Circuits with multiple inductors may be simplified by replacing a network of inductors with one equivalent inductor. As will be demonstrated in this section, the parallel and series networks may be replaced by their respective equivalent inductors in order to simplify the circuit.

❖ Serial Inductors

This section covers the process for simplifying a series network of inductors.



$$\text{Original Circuit} \rightarrow v = v_1 + v_2 + \dots + v_n \rightarrow v = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + \dots + L_n \frac{di(t)}{dt}$$

$$\text{Equivalent Circuit} \rightarrow v = L_{equ} \frac{di(t)}{dt}$$

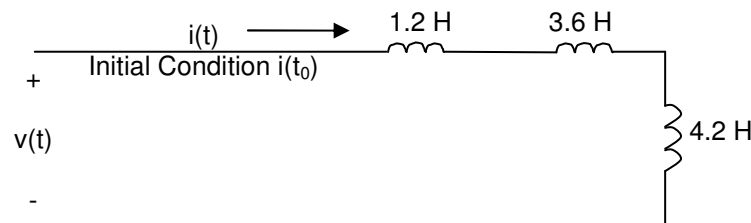
$$\text{From the above two equations} \rightarrow v = L_{equ} \frac{di(t)}{dt} = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + \dots + L_n \frac{di(t)}{dt}$$

$$L_{equ} \frac{di(t)}{dt} = (L_1 + L_2 + \dots + L_n)$$

$$\text{For Series network} \rightarrow L_{equ} = L_1 + L_2 + \dots + L_n \text{ with initial condition } i(t_0) \text{ at } t = t_0$$

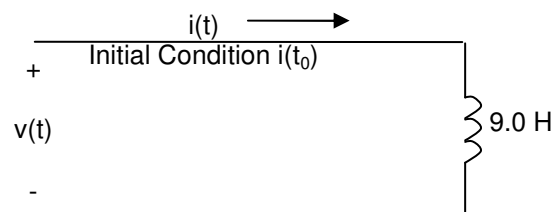
➤ Example – Simplify Series Inductors

Find the equivalent inductance of the circuit shown below:



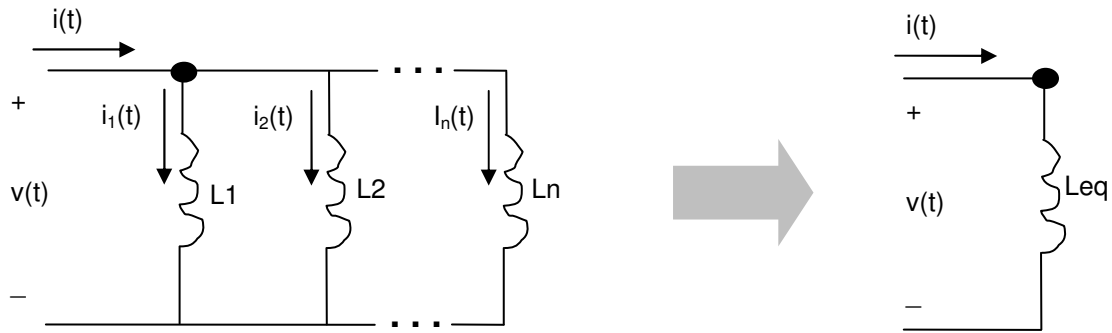
Solution

$$\text{Series network} \rightarrow L_{equ} = 1.2 + 3.6 + 4.2 = 9.0H$$



❖ Parallel Inductors

This section covers the process of simplifying a parallel inductor network.



KCL at the node in original circuit $\rightarrow i(t) = i_1(t) + i_2(t) + \dots + i_n(t)$

Note: $i_k(t)$ and $i_k(t_0)$ represent the instantaneous and initial condition current

$$i(t) = \frac{1}{L_1} \int v(t)dt + i_1(t_0) + \frac{1}{L_2} \int v(t)dt + i_2(t_0) + \dots + \frac{1}{L_n} \int v(t)dt + i_n(t_0)$$

$$i(t) = \left(\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right) \int v(t)dt + i_1(t_0) + i_2(t_0) + \dots + i_n(t_0)$$

KCL at the node in the equivalent circuit $\rightarrow i(t) = \frac{1}{L_{equ}} \int v(t)dt + i(t_0)$

From the above two equations \rightarrow

$$i(t) = \frac{1}{L_{equ}} \int v(t)dt + i(t_0) = \left(\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n} \right) \int v(t)dt + i_1(t_0) + i_2(t_0) + \dots + i_n(t_0)$$

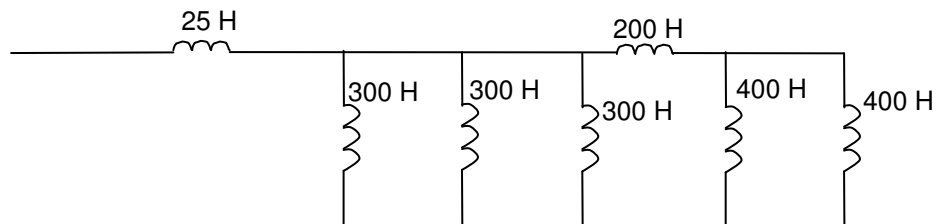
Therefore in a parallel inductors network, equivalent is calculated by:

$$\frac{1}{L_{equ}} = \frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n}$$

$$i(t_0) = i_1(t_0) + i_2(t_0) + \dots + i_n(t_0) \quad \text{"initial conditions of the resulting inductor"}$$

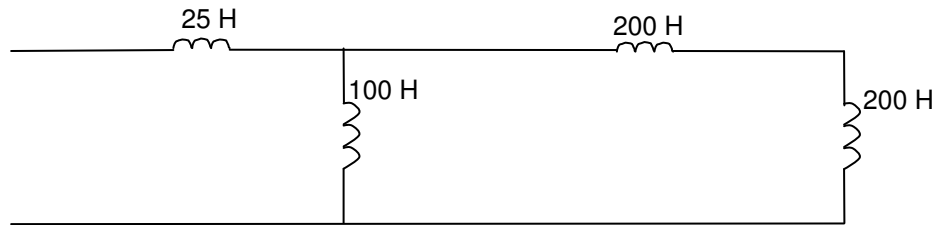
➤ Example – Simplification of Parallel Inductors Network

Simplify the following Circuit:

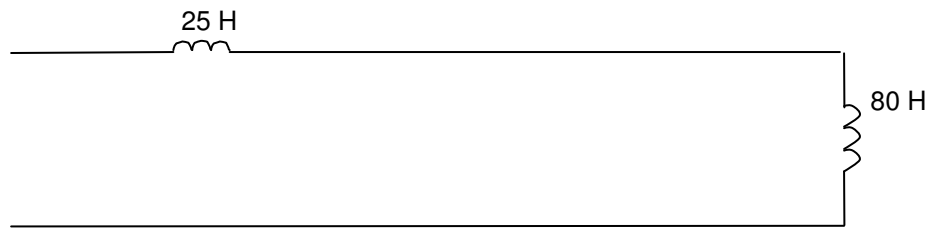


The three 300 H are in parallels $\rightarrow L_{equ1} = \frac{1}{\frac{1}{300} + \frac{1}{300} + \frac{1}{300}} = 100H$

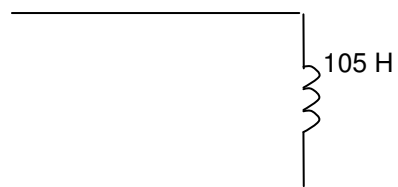
The two 400 H are in parallels $\rightarrow L_{equ2} = \frac{1}{\frac{1}{400} + \frac{1}{400}} = 200H$



The 100 H is in parallel with (200+200) H $\rightarrow L_{equ3} = \frac{1}{\frac{1}{100} + \frac{1}{400}} = 80H$



The equivalent of the total network $\rightarrow L_{eq} = 25 + 80 = 105 H$



6.4. Capacitor, C

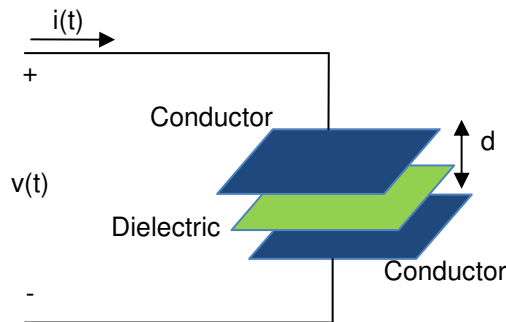
A capacitor is a circuit element made up of two electricity conducting surface, separated by dielectric (a non-conducting or insulating) material. Capacitors are manufactured from variety of material and are packaged based on the application requirement so it is difficult to know a capacitor characteristic without the manufacturer's specifications.

Below is a basic capacitor construction for the purpose of demonstrating the underlying physical phenomenon and corresponding symbol:

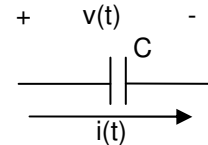
Physical Construction →

$$C = \frac{\epsilon A}{d} \text{ where}$$

C is capacitance in Farad
A is area of conductor plate
d is the thickness of dielectric
 ϵ the permittivity of the dielectric



Circuit Symbol →



As voltage varies with time, the displacement of charge also varies with time, causing what is known as the displacement current. The displacement current is proportional with the change in voltage over time and can be calculated based on the following equation:

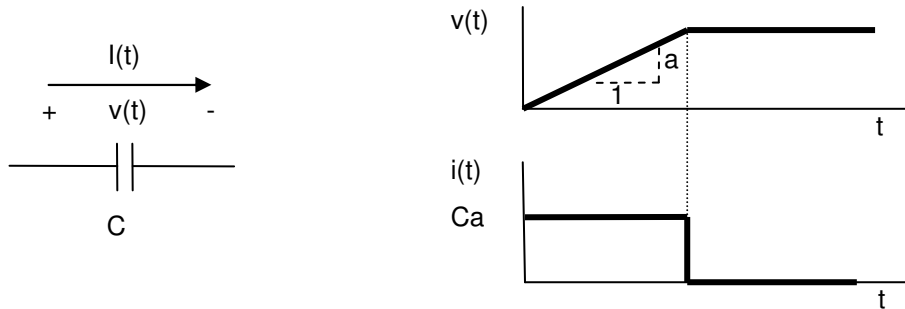
$$i(t) = C \frac{dv}{dt} \text{ where:}$$

v(t) is the voltage in Volts
i(t) is the current in Amps
C is measure of capacitance in Farad

Here a few observations based on this equation:

- If voltage, v(t), is constant then the current through the capacitor is zero.
- When the voltage changes instantly ($dt \rightarrow 0$) then the current, i(t) will be infinity ($i(t) \rightarrow \infty$). Although voltage cannot change instantly, the nearest approximates is to place a voltage source across the capacitor terminal long enough to fully charge the capacitor $\{i(t)=0\}$, then replace the source with a short across the terminals. This experiment will results in a large current discharge and may be extremely dangerous due to large instantaneous current generated.

- Capacitor is a passive component meaning it does not produce power but is able to store energy. Below is an example of relating a linear change in voltage with resulting current in a capacitor:



Integrating the earlier equations $i(t) = C \frac{dv}{dt}$ provides with an equation to calculate Voltage across the capacitor in term of current:

$$v(t) = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0)$$

Power and energy equations will complete the discussion of capacitor's behavior:

- Capacitor Power, $p(t)$ Calculation
Form earlier discussion → Start with $p(t) = v(t) * i(t)$
Substitute Capacitance current equation in the $p(t)$ equation →

$$p(t) = C v(t) \frac{dv}{dt} \text{ where:}$$

Power, $p(t)$, in Watts (W)
voltage, $v(t)$, in Volts (A)
Capacitance, C , in Farads (F)

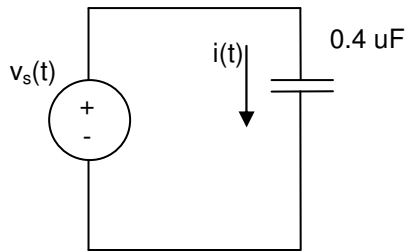
- Capacitor Energy or Work, $w(t)$, Calculation
Form earlier discussion → $p(t) = dw/dt \rightarrow dw/dt = Cvdv/dt \rightarrow dw = Cvdv$
Integrate both side with respect to $t \rightarrow$

$$w(t) = \frac{1}{2} C v^2(t) \text{ where:}$$

Energy or Work, $w(t)$ is in Joules
Voltage, $v(t)$ is in Volts
Capacitance, C , is in Farads

Finally it is important to notice the duality of Capacitors and Inductors as one is driven by voltage change while the other is driven by the current change.

- ❖ Example – Applying Basic Capacitance Relationships
For the following Circuit:



The voltage at the terminals of 0.4 uF capacitor is:

$$v_g(t) = 0 \text{ V} \quad t < 0$$

$$v_g(t) = 20e^{-5,000t} \sin 10,000t \text{ V} \quad t \geq 0$$

Find:

a) $i(t)$ at $t=0^+$

b) power delivered to the capacitor at $t=100 \text{ usec.}$

c) energy stored in the capacitor at $t=200 \text{ usec.}$

Solutions

a) Find $i(0^+)$

$$i(t) = C \frac{dv}{dt} = (0.4 * 10^{-6})(20 * (-5,000e^{-5,000t} \sin 10,000t + 10,000e^{-5,000t} \cos 10,000t))$$

$$i(0) = 8 * 10^{-6} * (0 + 10,000) = 0.08 \text{ A}$$

b) Find the power delivered to the capacitor at $t= 100 \text{ usec.}$

$$p(t) = i(t) * v(t)$$

$$p(t) = (8 * 10^{-6})(-5,000e^{-5,000t} \sin 10,000t + 10,000e^{-5,000t} \cos 10,000t)(20)(e^{-5,000t} \sin 10,000t)$$

$$p(t = 0.0001) = 0.059 \text{ w} = 59 \text{ mW}$$

c) Find the energy stored in the capacitor at $t=200 \text{ usec.}$

$$w(t) = \frac{1}{2} C v^2$$

$$w(t) = \frac{1}{2} (8 * 10^{-6})(-5,000e^{-5,000t} \sin 10,000t + 10,000e^{-5,000t} \cos 10,000t)^2$$

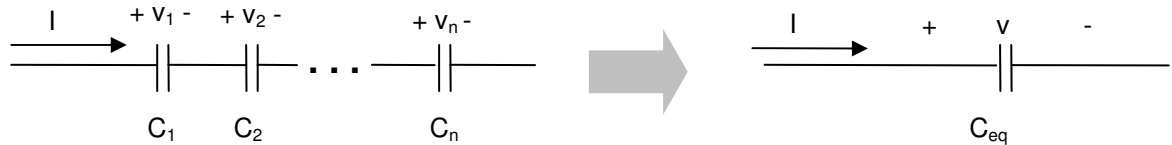
$$w(t = 0.0002) = 41.05 \text{ J}$$

6.5. Series and Parallel Capacitors

Circuits with multiple capacitors may be simplified by replacing a network of capacitors with one equivalent capacitor. As will be discussed in this section, the parallel and series networks may be replaced by their respective equivalent to simplify the circuit.

❖ Serial capacitor network

This section covers the calculation of equivalent capacitor for a series capacitor network.



Original Circuit $\rightarrow v = v_1 + v_2 + \dots + v_n$

$$\frac{1}{C_1} \int i(t) dt + \frac{1}{C_2} \int i(t) dt + \dots + \frac{1}{C_n} \int i(t) dt + v_1(t_0) + v_2(t_0) + \dots + v_n(t_0)$$

Equivalent Circuit $\rightarrow v = \frac{1}{C} \int i(t) dt + v(t_0)$

From the above two equations \rightarrow

$$\frac{1}{C_{equ}} \int i(t) dt + v(t_0) = \frac{1}{C_1} \int i(t) dt + \frac{1}{C_2} \int i(t) dt + \dots + \frac{1}{C_n} \int i(t) dt + v_1(t_0) + v_2(t_0) + \dots + v_n(t_0)$$

Therefore \rightarrow

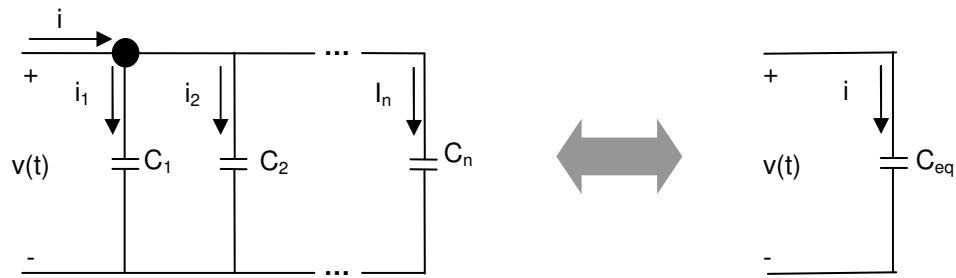
$$\frac{1}{C_{equ}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}$$

For series Capacitors

$$v_1(t_0) = v_1(t_0) + v_2(t_0) + \dots + v_n(t_0) \quad \text{"Initial condition relationship"}$$

❖ Parallel capacitor network

This section covers the calculation of equivalent capacitor for a parallel capacitor network.



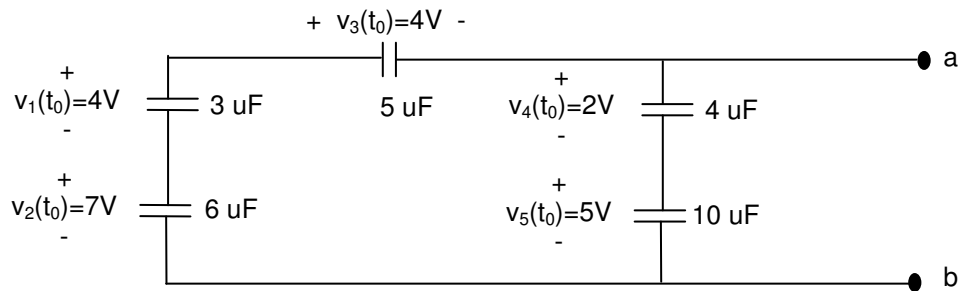
$$I = I_1 + I_2 + \dots + I_n$$

$$C_{eq} dv/dt = C_1 dv/dt + C_2 dv/dt + \dots \rightarrow$$

$$C_{eq} = C_1 + C_2 + C_3 + \dots \quad \text{for Parallel Capacitors}$$

❖ Example – Application of Capacitor Network Simplification

The following capacitor network at time t_0 is shown with the following initial condition. Find the equivalent capacitor for the following network:



Solutions

5, 3 and 6 uF capacitors are in series \rightarrow

$$C_{equA} = \frac{1}{\frac{1}{5} + \frac{1}{3} + \frac{1}{6}} = 1.4 \mu F$$

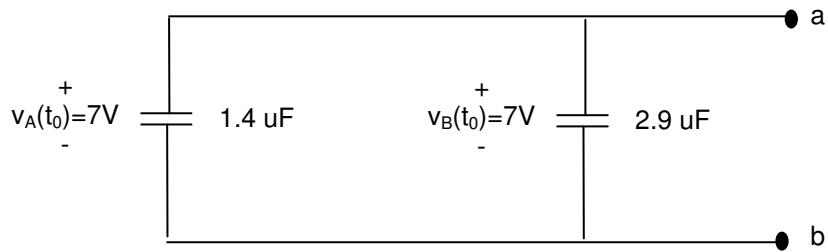
$$v_A(t_0) = v_1(t_0) + v_2(t_0) + v_3(t_0) = 7 + 4 - 4 = 7 \text{ V}$$

4 and 10 uF capacitors are in series \rightarrow

$$C_{equB} = \frac{1}{\frac{1}{4} + \frac{1}{10}} = 2.9 \mu F$$

$$v_B(t_0) = v_4(t_0) + v_5(t_0) = 7 \text{ V}$$

Redraw the circuit as:



1.4 and 2.9 uF capacitors are in parallel \rightarrow

$$C_{\text{eq}} = C_A + C_B = 1.4 + 2.9 = 4.3 \text{ uF}$$

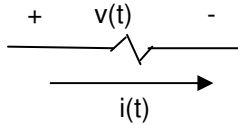
$$v_{\text{eq}}(t_0) = 7 \text{ V}$$



6.6. Summary

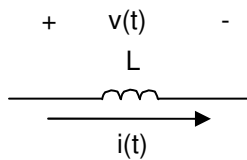
❖ Relationships

Resistor, R Ohms



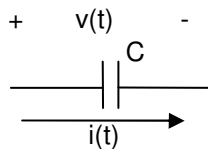
$$\begin{aligned} v(t) &= R * i(t) & i(t) &= \frac{v(t)}{R} \\ p(t) &= \frac{v(t)^2}{R} = i(t)^2 R & w(t) &= \int_{t_0}^t p(\tau) d\tau + w(t_0) \end{aligned}$$

Inductor, L Henrys



$$\begin{aligned} v(t) &= L \frac{di(t)}{dt} & i(t) &= \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + I(t_0) \\ p(t) &= Li(t) \frac{di(t)}{dt} & w(t) &= \frac{1}{2} Li^2(t) \end{aligned}$$

Capacitor, C Farads



$$\begin{aligned} i(t) &= C \frac{dv}{dt} & v(t) &= \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0) \\ p(t) &= C v(t) \frac{dv}{dt} & w(t) &= \frac{1}{2} C v^2(t) \end{aligned}$$

❖ Simplification

Resistors

Series: $R_{equ} = R_1 + R_2 + \dots + R_n$

Parallel: $\frac{1}{R_{equ}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}$

Inductors

Series: $L_{equ} = L_1 + L_2 + \dots + L_n$

$i_{equ}(t_0) = i_1(t_0) = i_2(t_0) = \dots = i_n(t_0)$

Parallel: $\frac{1}{L_{equ}} = \frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n}$

$i_{equ}(t_0) = i_1(t_0) + i_2(t_0) + \dots + i_n(t_0)$

Capacitors

Series: $\frac{1}{C_{equ}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}$

$V_{equ}(t_0) = V_1(t_0) = V_2(t_0) = \dots = V_n(t_0)$

Parallel: $C_{equ} = C_1 + C_2 + \dots + C_n$

$V_{equ}(t_0) = V_1(t_0) = V_2(t_0) = \dots = V_n(t_0)$

6.7. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 6.

6.8. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 7. First-Order (RL & RC) Circuits

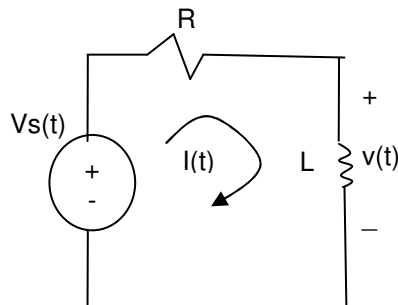
Key Concepts and Overview

- ❖ Definitions
- ❖ Step Response of a First Order Circuit (RL & RC)
- ❖ Natural (Source-Free) of a First Order Circuit (RL & RC)
- ❖ Additional Resources

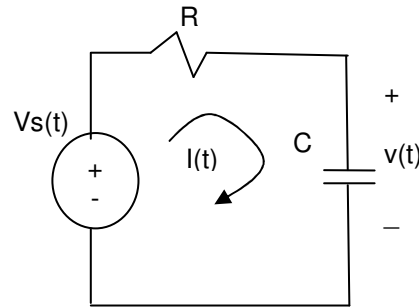
7.1. Definitions

Inductors (L) and capacitors (C) have the ability to store energy which means the relationships between current and voltage in circuits with L and C are not linear algebraic as they were with only resistors (R). Therefore, analysis of circuits with R, L and C will be discussed in two levels:

First Order Circuit – The circuit with only one storage element (C or L) and R is called first order circuit since the equation describing the relationship between current and voltage is first order differential equation. The first order circuit may be a RL circuit or a RC circuit. This chapter introduces first order circuits. Instantaneous equation for given RL and RC circuits may be written as a differential equation and solved as shown below:



$$-V_s + Ri + L \frac{di}{dt} = 0$$

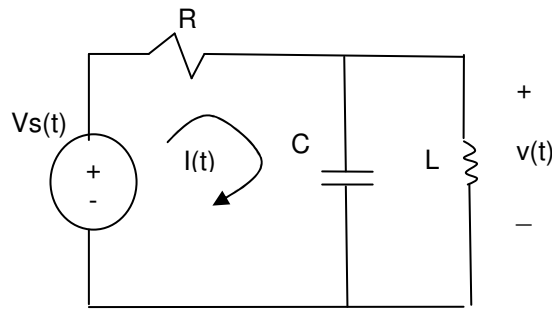


$$-v^s + Ri + \frac{1}{C} \int i dt = 0$$

or

$$-\frac{dv_s}{dt} + R \frac{di}{dt} + \frac{1}{C} i = 0$$

Second Order Circuit – The circuits with both L and C present is a second order circuit which is also refers to as an RLC circuit. This is called second order circuit since the equation describing the relationship between current of voltage is 2nd order differential equation. This type of circuit will be covered in the next chapter. Instantaneous equation for a given RLC circuit may be written as a differential equation using KCL as shown below:



$$\frac{(v - v_s)}{R} + C \frac{dv}{dt} + \frac{1}{L} \int v dt = 0$$

or

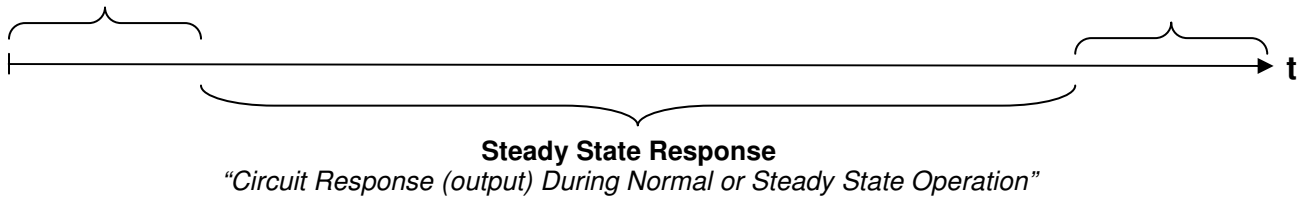
$$\frac{1}{R} \frac{d(v - v_s)}{dt} + C \frac{d^2 v}{dt^2} + \frac{1}{L} v = 0$$

For any circuit with storage elements (C & L), finding the circuit response (output) requires understanding the three distinct circuit phases (Start up, on-going operation and shutdown). The following three responses match the three phases of a typical RC, RL or RLC circuit:

Step Response - Refers to circuit analysis and response when the power is first applied and before the circuit has stabilized. The Step Response for first order circuits (RC and RL) will be discussed in the next sections.

Steady State Response - After the circuit has been powered long enough for the circuit to stabilize and long enough before the power is removed as not to impact operation, it is said that the circuit is in Steady State. The circuit analysis in this phase is referred to as the Steady State analysis and the output is referred to as the steady state response. During this state, the circuit is operating in Steady State mode.

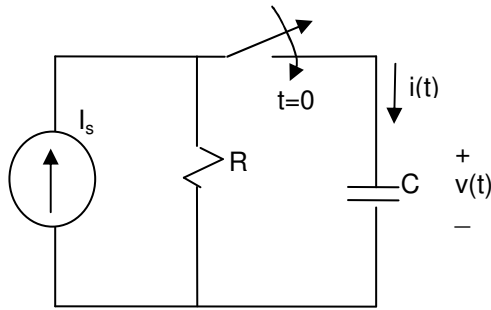
Natural Response - After the circuit power supply has been removed, the circuit is said to be in Natural state (no external power). The Response from this type of circuit is referred to as the natural response. The Natural Response for first order circuits (RC and RL) will be discussed in this chapter also.

Step Response*"Circuit Response During Power Up"***Natural Response***"Circuit Response During Power Down"*

The remainder of this chapter introduces analysis of Step Response and Natural Response of RL and RC circuits and associated first order differential equations. Next chapter will repeat the analysis for RLC circuits and associated second order differential equations. Finally, Chapter 9 covers the Steady State Response.

7.2. Step Response of a First Order Circuit (RL & RC)

Step response is the current and voltage that results when a source is connected to a circuit that has not been powered long enough so that the energy stored in the inductor and capacitor is equal to zero. In other words, the voltage across the capacitor equals zero and current through the inductor equals zero at the time the power is connected to the circuit. The following diagrams are typically used to demonstrate this concept.



Typical RC Circuit

Step Response ($t \geq 0$)

Use Node Voltage and integrate:

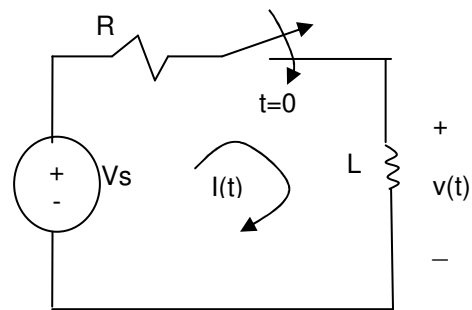
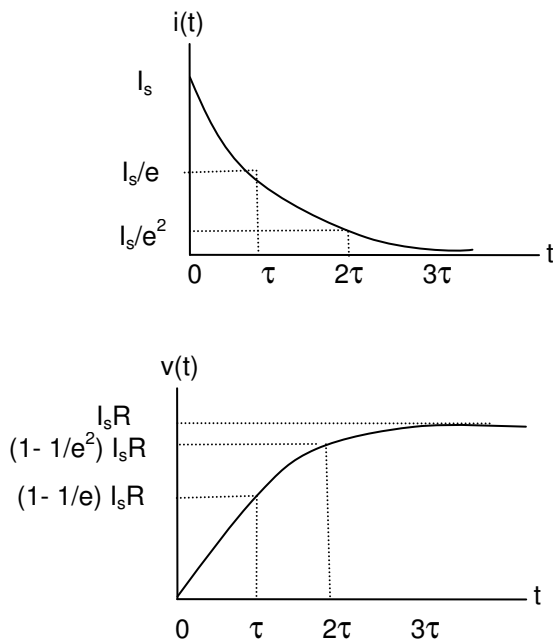
$$C \frac{dv}{dt} + \frac{v}{R} = I_s$$

$$v(t) = I_s R + (V_0 - I_s R) e^{-t/RC} \text{ for } t \geq 0$$

Where:

V_0 is initial capacitor current ($=0$)

Circuit Time Constant $\tau = RC$



Typical RL Circuit

Step Response ($t \geq 0$)

Use Mesh Current and Integrate:

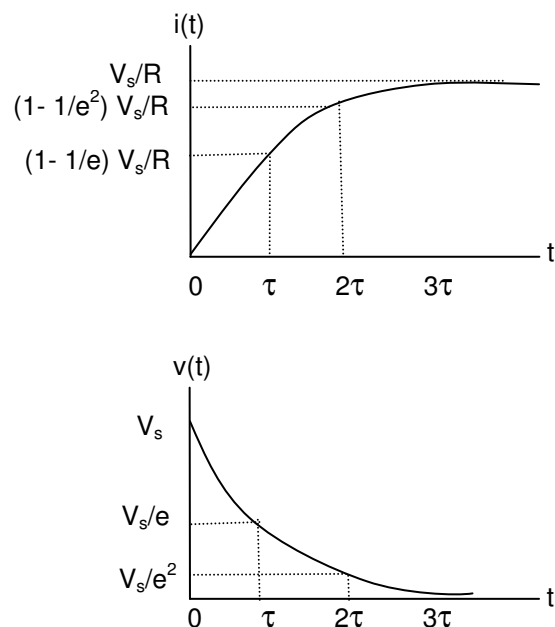
$$V_s = iR + L \frac{di}{dt}$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R} \right) e^{-(R/L)t} \text{ for } t \geq 0$$

Where:

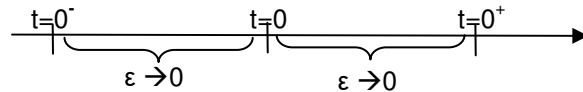
I_0 is initial inductor current ($=0$)

Circuit Time Constant $\tau = L/R$



❖ Observations

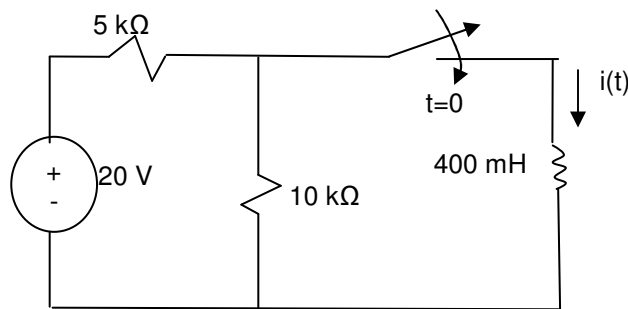
- As the power is applied to the capacitor, initially, it appears as a short and then after a long time ($\gg \tau$) it will appear as an open.
- As the power is applied to the inductor, initially, it appears as an open and then after a long time ($\gg \tau$) it will appear as a short.
- Typically $t=0^-$ is used to refer time just before 0 and $t=0^+$ is used to refer time just after time 0.



- Although the typical RL and RC circuits shown above are simple, from earlier chapters, it is understood that any first order circuit may be simplified to the typical circuit form:
 - Any RL circuit may be simplified to the typical form shown, by finding the venin's equivalent with respect to inductor's terminals.
 - Any RC circuit may be simplified to the typical form shown above by finding the Norton equivalent circuit with respect to capacitor's terminals.

❖ Examples – First Order Circuit Step Response

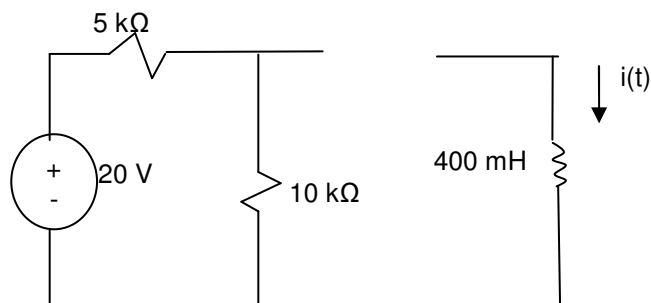
For the following circuit:



Switch has been open for a long time before the switch is closed at $t=0$. Calculate the value of $i(0^-)$, $i(0^+)$. Find the equations for $i(t)$ and $v(t)$ for $t > 0$.

Solution

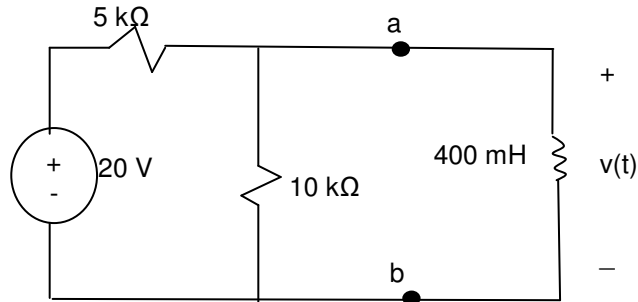
$t=0^-$ refers to time just before the switch is closed or $T \rightarrow 0$ from the negative side. $v(0^-)$ refer to voltage after the circuit has been in the following configuration for a long time.



Which means the inductor does not have any energy $\rightarrow i(0^-) = 0$ V

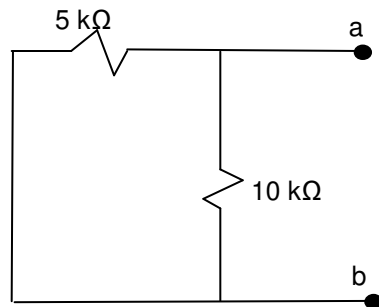
$t=0^+$ refers to time just after the switch is closed since $\Delta t = \{(t=0^+) - (t=0^-)\} \rightarrow 0$ and it physically not possible for inductor current to change instantly, we can conclude that $i(0^+) = i(0^-) = 0$ V

To Calculate $i(t)$ for $t>0$, draw the circuit with the switch closed:



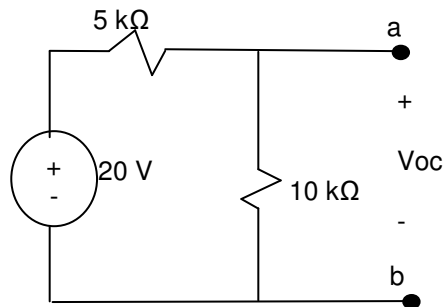
At this point we have a choice to either derive the differential equation for this circuit or get the circuit to the typical circuit which we have the solution for. The easier method is to find the Thevenin equivalent of the circuit with respect to terminals a and b.

R_{th} can be found by deactivating the voltage source (short) which reduces the circuit to:



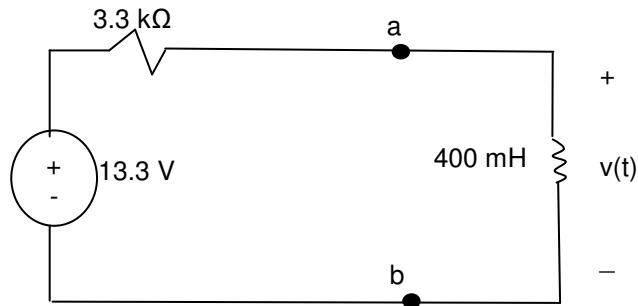
$$R_{th} = (5 \parallel 10) = 3.3 \text{ k}\Omega$$

To find V_{th} or V_{oc} , use the following circuit which is simply a voltage divider.



$$V_{oc} = 20 * 10 / (10 + 5) = 13.3 \text{ V}$$

Now that we have Thevenin equivalent, apply it to the original circuit at $t>0$ which results in the following equivalent circuit



With the above circuit, the Step Response equation derived for typical RL circuit can be used in this problem:

$$V_s = iR + L \frac{di}{dt}$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R} \right) e^{-(R/L)t} \text{ for } t \geq 0$$

$$i(t) = \frac{13.3}{3.3} + \left(0 - \frac{13.3}{3.3} \right) e^{-(3300 / 0.4)t} \text{ for } t \geq 0$$

$$i(t) = 4 - 4e^{-8250t} \text{ A for } t \geq 0$$

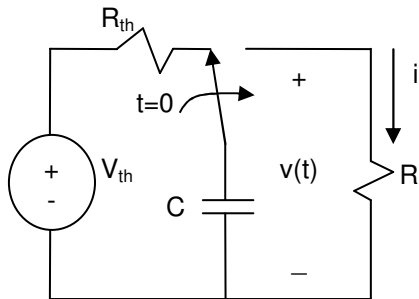
$$v(t) = L \frac{di(t)}{dt} = 0.4 * 4 * 8250 e^{-8250t} \text{ for } t \geq 0$$

$$v(t) = 19300 e^{-8250t} \text{ V for } t \geq 0$$

7.3. Natural (Source-Free) Response of a First Order Circuit (RL & RC)

When the circuit has been supplied by a source for a long-time such that the circuit is in steady state and then the source is removed, the response is called a Natural Response. In other words, Natural response is the currents and voltages that result from discharge of stored energy in the inductor and capacitors.

The following section uses a typical RL and RC circuit to derive the relationship for Natural Response.



Typical RC Circuit

Natural Response ($t \geq 0$)

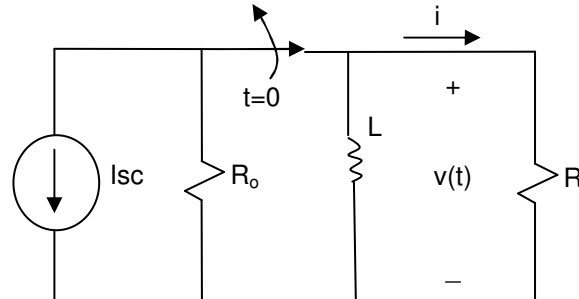
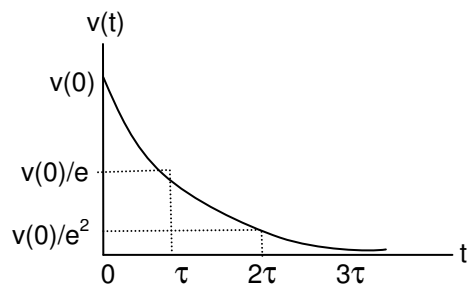
Using Node Voltage & Integrate:

$$C \frac{dv}{dt} + \frac{v}{R} = 0$$

$$v(t) = v(0)e^{-t/RC} \text{ for } t \geq 0$$

Where:

Circuit Time Constant $\tau = RC$



Typical RL Circuit

Natural Response ($t \geq 0$)

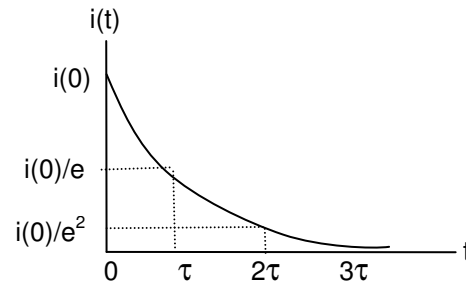
Using Mesh Current & Integrate:

$$L \frac{di}{dt} + iR = 0$$

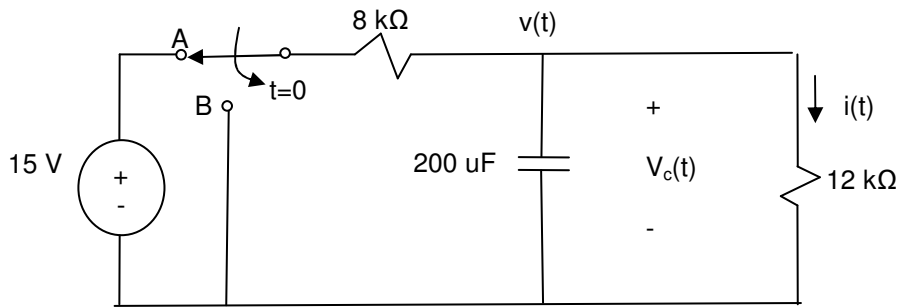
$$i(t) = i(0)e^{-(R/L)t} \text{ for } t \geq 0$$

Where:

Circuit Time Constant $\tau = L/R$



- ❖ Examples – First Order Circuit Natural Response
For the following circuit:

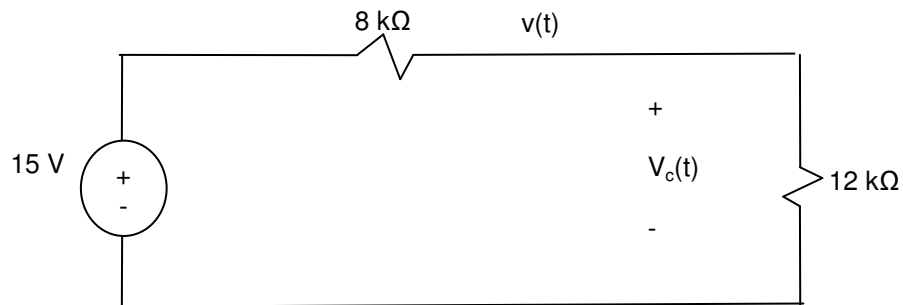


V

Switch has been in position A for a long time. At the time $t=0$, the switch is moved to position B, find $i(t)$ for $t > 0$.

Solution

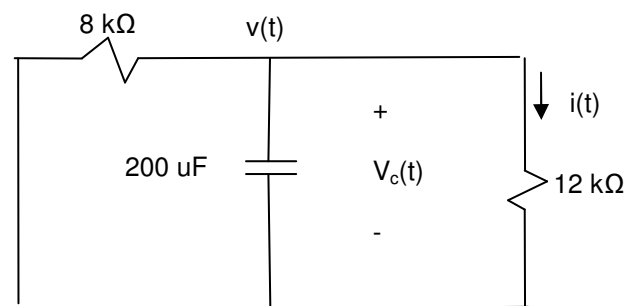
First calculate the Initial condition $v_c(0^-)$. C has been energized for a long time, therefore the C will appear open. The circuit can be redrawn as shown below just before the switch is moved:



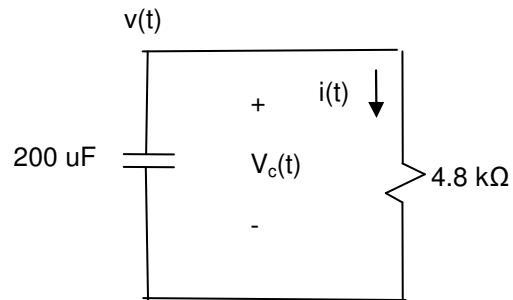
$$v_c(0^-) = 15 \cdot 12 / (8 + 12) = 9 \text{ V} \text{ Initial voltage of the capacitor.}$$

At time $t > 0$, after the switch has been moved to Position B →

Our goal is to simplify the circuit to be the same as the typical RC circuit so that we can apply the equations derived.



Combine the resistors →



$$\text{KCL} \rightarrow 0.2 \frac{dv_c}{dt} + \frac{v_c}{4800} = 0$$

Apply the natural response equation $C \frac{dv}{dt} + \frac{v}{R} = 0$

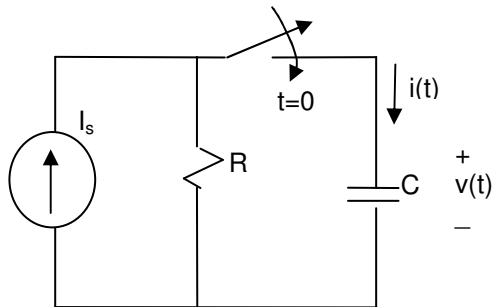
$$v(t) = v(0)e^{-t/RC} \text{ for } t \geq 0$$

$$v(t) = 9e^{-t/.96} \text{ for } t \geq 0$$

$$i(t) = \frac{v(t)}{R} = \frac{9e^{-t/.96}}{4800} \text{ for } t \geq 0$$

7.4. Summary

❖ Step Response of a First Order Circuit (RL & RC)



Typical RC Circuit

Step Response ($t \geq 0$)

Use Node Voltage and integrate:

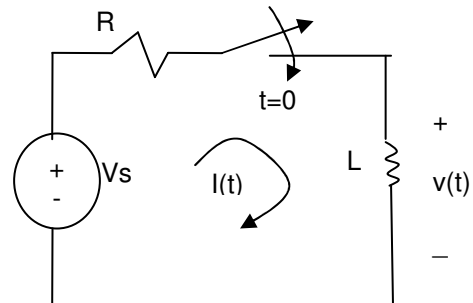
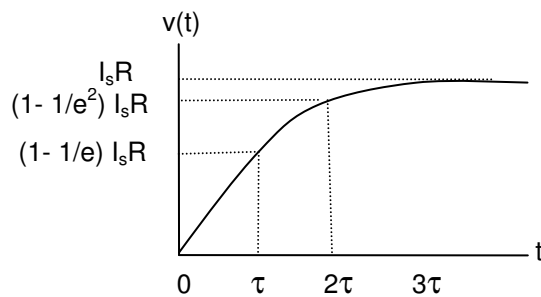
$$C \frac{dv}{dt} + \frac{v}{R} = I_s$$

$$v(t) = I_s R + (V_0 - I_s R) e^{-t/RC} \text{ for } t \geq 0$$

Where:

V_0 is initial capacitor voltage ($=0$)

Circuit Time Constant $\tau = RC$



Typical RL Circuit

Step Response ($t \geq 0$)

Use Mesh Current and Integrate:

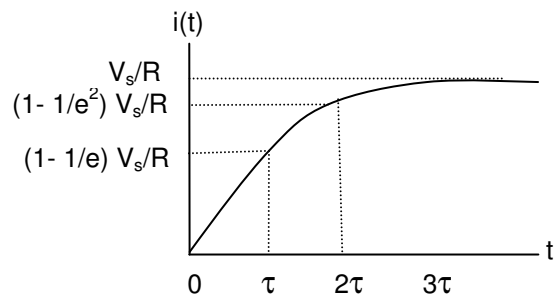
$$V_s = iR + L \frac{di}{dt}$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R} \right) e^{-(R/L)t} \text{ for } t \geq 0$$

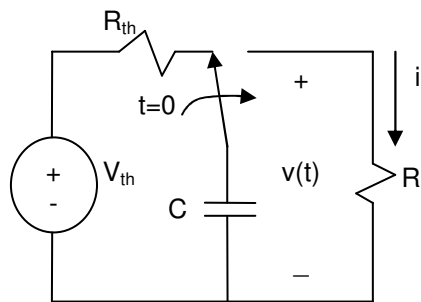
Where:

I_0 is initial inductor current ($=0$)

Circuit Time Constant $\tau = L/R$



❖ Natural Response of a First Order Circuit (RL & RC)



Typical RC Circuit

Natural Response ($t \geq 0$)

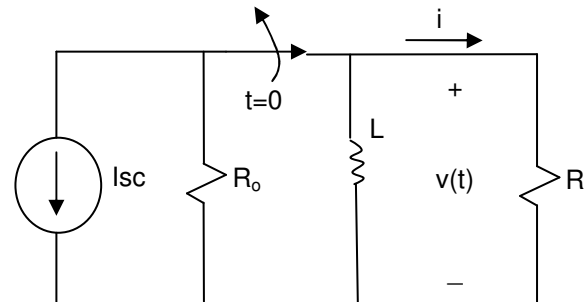
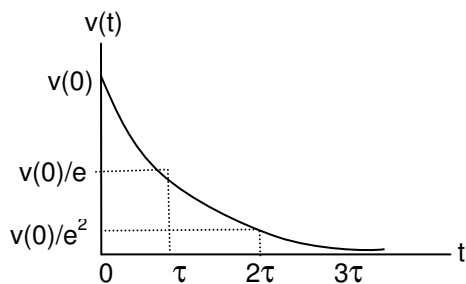
Using Node Voltage & Integrate:

$$C \frac{dv}{dt} + \frac{v}{R} = 0$$

$$v(t) = v(0)e^{-t/RC} \text{ for } t \geq 0$$

Where:

Circuit Time Constant $\tau = RC$



Typical RL Circuit

Natural Response ($t \geq 0$)

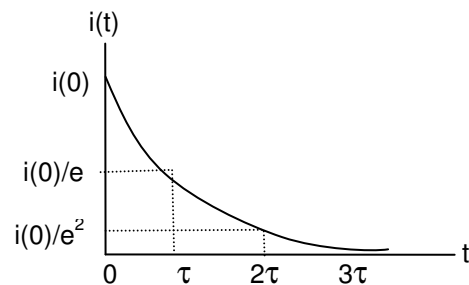
Using Mesh Current & Integrate:

$$L \frac{di}{dt} + iR = 0$$

$$i(t) = i(0)e^{-(R/L)t} \text{ for } t \geq 0$$

Where:

Circuit Time Constant $\tau = L/R$



❖ Process of determining Step and Natural Response

- 1) Decide if the circuit has Step Response (includes at least one source) after the change or the circuit has Natural Response (No Source). Select the appropriate typical circuit & associated solution.
- 2) Calculate initial conditions (After a long time of constant voltage and current settings). (Inductor is short or $v_L(t_0)=0$; Capacitor is open or $i_C(t_0)=0$)
- 3) Use the standard solution form for your circuit to calculate $v(t)$ and $i(t)$ for $t > t_0$.

7.5. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 7.

7.6. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 8. Second Order (RLC) Circuits

Key Concepts and Overview

- ❖ Overview
- ❖ Parallel RLC Circuit Step Responses
- ❖ Parallel RLC Circuit Natural Responses
- ❖ Series RLC Circuit Step Responses
- ❖ Series RLC Circuit Natural Responses
- ❖ Summary – General Form
- ❖ Additional Resources

8.1. Overview

This chapter is focused on RLC circuit which includes at least a capacitor, an inductor and a resistor. Techniques such as KCL, KVL and their extensions from earlier chapters will be used to analyze the RLC circuit. The resulting equations for RLC circuit step and natural responses will be ordinary second order differential equations. The word ordinary implies the all derivatives are with respect to one variable which is time, t .

It is understood that not all ordinary differential equations have analytical solution (mathematically solvable). In many cases numerical techniques and aid of computer programming will be required to approximate the answer. In this chapter the analyzed circuits will be limited to pure series or pure parallel RLC circuits which ensure the resulting differential equations are of the following form:

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = 0$$

This differential equation form is known to have analytical solutions consisting of linear combinations of Ke^{st} . Differential equation solutions are typically found by using the known differential equations and solutions form. The specific solutions to a differential equation can be found by plugging the known archetype solutions into to differential equation to find the specific solutions. The initial condition values of $x(t)$ and dx/dt are used to find the associated constants and parameters.

The following derivation uses the technique described above to find the specific solutions to the earlier general differential equation using the archetype solutions, $x(t)=Ke^{st}$:

$$Ks^2 e^{st} + 2\alpha Ks e^{st} + \omega_0^2 K e^{st} = 0$$

Ke^s is factored out.

$$Ke^{st} (s^2 + 2\alpha s + \omega_0^2) = 0$$

It is known that $Ke^{st} \neq 0$ for all finite values of t . Therefore, both sides can be divided by Ke^{st} , resulting in the following quadratic equation:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Solving the above quadratic equation \rightarrow

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Therefore response, $x(t)$, may be written as:

$$x(t) = X_f + K_1 e^{s_1 t} + K_2 e^{s_2 t} \text{ Where:}$$

Characteristic equation is $\{s^2 + 2\alpha s + \omega_0^2 = 0\}$

s_1 and s_2 are the characteristic roots, also referred to as the complex frequencies. They describe the mathematical character of the system.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \text{ rad / sec}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \text{ rad / sec}$$

α is the neper frequency, also referred to as the damping coefficient or factor. The name is derived from the fact that $x(t)$ is damped equal to $e^{-\alpha}$.

ω_0 is the resonant radian frequency, also referred to as the undamped natural frequency.

ω_d is the **damped radian frequency**, and is defined as: $w_d = \sqrt{w_0^2 - \alpha^2}$ rad/sec

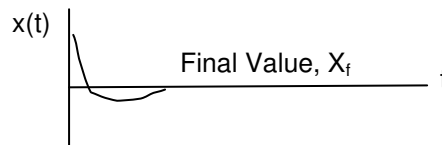
X_f is the time independent portion of $x(t)$ and also referred to as the final value. The name “final value” means that as t approaches infinity, X_f is the only non-zero portion of $x(t)$. In other words, $X_f = x(\infty)$.

K_1 and K_2 are constants that are calculated using the initial condition of $x(t)$ and dx/dt .

Depending on the relationship between ω_0 and α , the response, $x(t)$, may be Overdamped ($\omega_0^2 < \alpha^2$), Critically damped ($\omega_0^2 = \alpha^2$), or Underdamped ($\omega_0^2 > \alpha^2$). More information on each of the three circuit types are presented below:

➤ **$x(t)$ is Overdamped if $\omega_0^2 < \alpha^2$**

- s_1 and s_2 will be both real and distinct.
- $x(t)$ approaches final value without oscillation

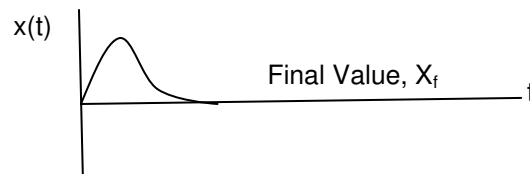


- Response may be rewritten as:

$$x(t) = X_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

➤ **$x(t)$ is Critically damped if $\omega_0^2 = \alpha^2$**

- s_1 and s_2 will be both real and equal.
- $x(t)$ is on the verge of Oscillating about its final value

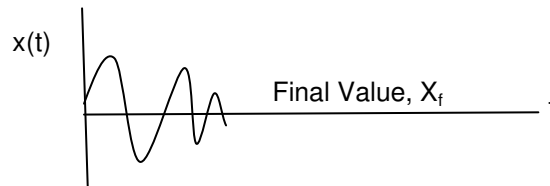


- Response may be rewritten as:

$$x(t) = X_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

➤ **$x(t)$ is Underdamped if $\omega_0^2 > \alpha^2$**

- s_1 and s_2 will be both complex and conjugate of each other.
Note: $(a+jb)$ is conjugate of $(a-jb)$; conjugate means j is replaced by $-j$.
- $x(t)$ oscillates about its final value



- Response may be rewritten as:

$$x(t) = X_f + B_1 e^{-\alpha t} \cos w_d t + B_2 e^{-\alpha t} \sin w_d t$$

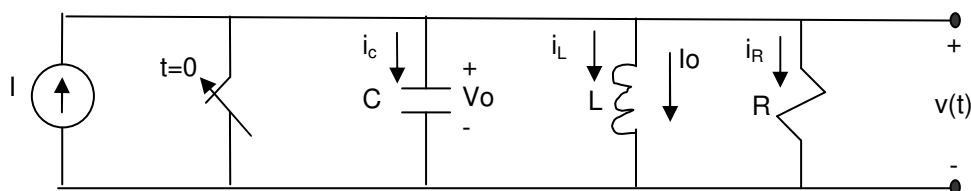
The remainder of this chapter focuses on application of the above mathematical model to Step and Natural responses analysis of the following specific circuit types:

1. Parallel RLC Circuit Step Response
2. Parallel RLC Circuit Natural Response
3. Series RLC Circuit Step Response
4. Series RLC Circuit Natural Response

The steady state RLC circuit response will be covered in chapter 9.

8.2. Step Response of a Parallel RLC Circuit

This section introduces the Parallel RLC circuit and step response analysis. At time, $t=0$, the switch is opened which means current starts to flow to the RLC part of the circuit. For this analysis, it is assumed that energy stored in C as represented by voltage, V_0 and the energy in L as represented by current, I_0 , are both equal to zero at $t=0$.



Apply KCL →

$$i_C + i_L + i_R = I$$

$$C \frac{dv}{dt} + \frac{1}{L} \int_0^t v dt + \frac{v}{R} = I$$

differentiate and re-arrange

$$\frac{d^2 v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{1}{LC} v = 0$$

Comparing the above equation with the characteristic equation: $\frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = 0 \rightarrow$

$$\alpha = \frac{1}{2RC} \text{ rad/sec} \quad \text{Neper Frequency}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ rad/sec} \quad \text{Resonant Radian Frequency}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \text{ rad/sec} \quad \text{Damped Radian Frequency}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 1}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 2}$$

Depending on the relationship between ω_0 and α , the response “output voltage, $v(t)$ ” may be Overdamped ($\omega_0^2 < \alpha^2$), Underdamped ($\omega_0^2 > \alpha^2$) or Critically damped ($\omega_0^2 = \alpha^2$), one of the following three types:

$$\text{Overdamped} (\omega_0^2 < \alpha^2) \Rightarrow v_C(t) = V_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\text{Critically Damped} (\omega_0^2 = \alpha^2) \Rightarrow v_C(t) = V_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$\text{Underdamped} (\omega_0^2 > \alpha^2) \Rightarrow v_C(t) = V_f + B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$\text{Overdamped}(\omega_0^2 < \alpha^2) \Rightarrow i_L(t) = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$\text{Critically Damped}(\omega_0^2 = \alpha^2) \Rightarrow i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

$$\text{Underdamped}(\omega_0^2 > \alpha^2) \Rightarrow i_L(t) = I_f + B_1' e^{-\alpha t} \cos w_d t + B_2' e^{-\alpha t} \sin w_d t$$

Where:

- $A_1, A_2, B_1, B_2, D_1, D_2, A_1', A_2', B_1', B_2', D_1', D_2'$ coefficients can be found by using the initial conditions:
 - For voltage use $v(0^+) = V_0$ and $dv(0^+)/dt$ which are derived from the capacitor voltage.
 - For current use $i(0^+) = I_0$ and $di(0^+)/dt$ which are derived from the inductor current.
- V_f is the value of voltage source or the capacitor voltage as t approaches infinity.
- I_f is the value of current source or the inductor current as t approaches infinity.

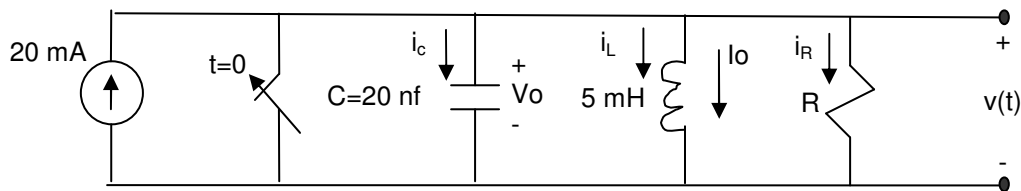
In summary, the following four steps may be used to find $v(t)$:

- (1) Find roots of Characteristic equation using R, L & C value.
- (2) Decide if the circuit is over damped, under damped or critically damped.
- (3) Find $v(0^+)$ and $dv(0^+)/dt$ using circuit analysis
- (4) Find Coefficients of $v(t)$ using initial Conditions.

Similar approach may be used to find the expression for $i(t)$. Additionally, this process is applicable to all 4 types of circuits discussed in the chapter.

❖ Example – Parallel RLC Step Response

For the following circuit where the switch is opened after a long time.:



Find:

- Range of R values such that the circuit is underdamped, overdamped and critically damped.
- Expression for $i_L(t)$ when $t \geq 0$ and $R=250 \Omega$.

Solution

- Range of R values such that the circuit is underdamped, overdamped and critically damped

$$\alpha = \frac{1}{2RC} = \frac{25 \times 10^6}{R} \text{ rad/sec} \quad \text{Neper Frequency}$$

$$w_0 = \sqrt{\frac{1}{LC}} = 10^5 \text{ rad/sec} \quad \text{Resonant Radian Frequency}$$

Conditions for each type of circuit response are listed below:

$$\text{Overdamped}(\omega_0^2 < \alpha^2) \Rightarrow 10^{10} < \frac{625 \times 10^{12}}{R^2} \Rightarrow R^2 < 625 \times 10^2 \Rightarrow R < 250\Omega$$

$$\text{Critically Damped}(\omega_0^2 = \alpha^2) \Rightarrow 10^{10} = \frac{625 \times 10^{12}}{R^2} \Rightarrow R = 250\Omega$$

$$\text{Underdamped}(\omega_0^2 > \alpha^2) \Rightarrow 10^{10} > \frac{625 \times 10^{12}}{R^2} \Rightarrow R^2 > 625 \times 10^2 \Rightarrow R > 250\Omega$$

b) Expression for $i_L(t)$ when $t \geq 0$ and $R=250\Omega$

Based on Calculation in part a, the circuit is critically damped for this value of R. Therefore, the Critically damped current equation for the circuit may be used:

$$\text{Critically Damped}(\omega_0^2 = \alpha^2) \Rightarrow i_L(t) = I + D_1' te^{-\alpha t} + D_2' e^{-\alpha t}$$

I is the final value of Inductor current which would be the same as the source current of 20 mA therefore:

$$i_L(t) = 0.02 + D_1' te^{-100000t} + D_2' e^{-100000t}$$

Now, need to find the values of D_1' and D_2' using the initial conditions

There is no energy stored in inductor prior to opening the switch $\rightarrow i_L(0^+) = 0$
initial voltage of the capacitor is zero therefore it will after switch is closed

$$v = \frac{L di_L}{dt} \Rightarrow \frac{di_L}{dt}(0^+) = 0$$

Apply the initial conditions to the characteristic equation.

$$i_L(0^+) = 0.02 + D_2' = 0 \Rightarrow D_2' = -0.02$$

$$\frac{di_L}{dt}(0^+) = D_1' e^{-100000t} + D_1' t(-10^5) e^{-100000t} + D_2' (-10^5) e^{-100000t} = 0 \Rightarrow D_1' = 2,000$$

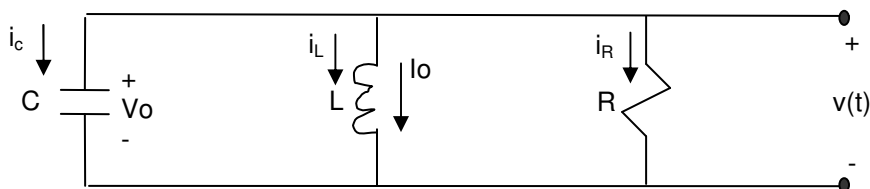
therefore

$$i_L(t) = 0.02 + 2,000e^{-100000t} - 0.02te^{-100000t}$$

It is recommended that the reader plot the $i_L(t)$ to see an example of critically damped signal.

8.3. Natural Response of a Parallel RLC Circuit

This section introduces the Parallel RLC circuit natural response analysis. At $t=0$, the source is removed from the circuit which means the energy stored in L and C will be powering the circuit. For this analysis, it is assumed that energy stored in C as represented by V_o and the energy in L as represent by I_o .



$$i_C + i_L + i_R = 0$$

$$C \frac{dv}{dt} + \frac{1}{L} \int_0^t v dt + I_o + \frac{v}{R} = 0$$

differentiate and re-arrange

$$\frac{d^2 v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{1}{LC} v = 0$$

Comparing the above equation to the characteristic equation: $\frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = 0 \rightarrow$

$$\alpha = \frac{1}{2RC} \text{ rad/sec} \quad \text{Neper Frequency}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ rad/sec} \quad \text{Resonant Radian Frequency}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \text{ rad/sec} \quad \text{Damped Radian Frequency}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 1}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 2}$$

Depending on the relationship between ω_0 and α , the response “output voltage, $v(t)$ ” may be Overdamped ($\omega_0^2 < \alpha^2$), Underdamped ($\omega_0^2 > \alpha^2$) or Critically damped ($\omega_0^2 = \alpha^2$), one of the following three types:

$$\text{Overdamped} (\omega_0^2 < \alpha^2) \Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\text{Critically Damped} (\omega_0^2 = \alpha^2) \Rightarrow v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$\text{Underdamped} (\omega_0^2 > \alpha^2) \Rightarrow v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$\text{Overdamped}(\omega_0^2 < \alpha^2) \Rightarrow i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\text{Critically Damped}(\omega_0^2 = \alpha^2) \Rightarrow i_L(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

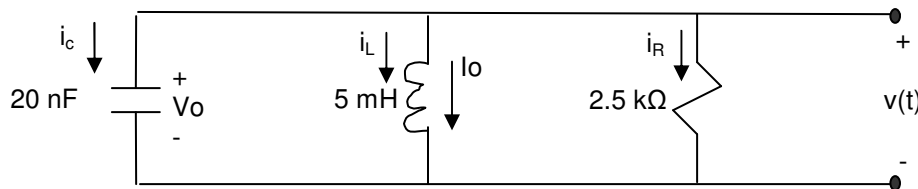
$$\text{Underdamped}(\omega_0^2 > \alpha^2) \Rightarrow i_L(t) = B_1 e^{-\alpha t} \cos w_d t + B_2 e^{-\alpha t} \sin w_d t$$

Where:

- $A_1, A_2, B_1, B_2, D_1, D_2, A_1', A_2', B_1', B_2', D_1', D_2'$ coefficients can be found by using the initial conditions:
 - For voltage use $v(0^+) = V_0$ and $dv(0^+)/dt$ which are derived from the capacitor voltage.
 - For current use $i(0^+) = I_0$ and $di(0^+)/dt$ which are derived from the inductor current.

❖ Example – Parallel RLC Natural Response

For the following circuit, source has been removed at $t=0$ after a long time of being connected.



a) Determine the response type.

b) Find the equation for $v(t)$ during $t > 0$ with initial conditions $V_0 = 0$ and $I_0 = -10$ mA at $t = 0$.

Solution

a) Determine the response type

$$\alpha = \frac{1}{2RC} = \frac{25 \times 10^6}{2500} = 10^4 \text{ rad/sec} \quad \text{Neper Frequency}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = 10^5 \text{ rad/sec} \quad \text{Resonant Radian Frequency}$$

Conditions for each type of circuit response are listed below:

Since $\omega_0^2 > \alpha^2 \Rightarrow \text{Underdamped circuit}$

b) Find the equation for $v(t)$ during $t > 0$ with initial conditions $V_0 = 0$ and $I_0 = -10$ mA at $t = 0$.

Since $\omega_0^2 > \alpha^2 \Rightarrow \text{Underdamped circuit} \Rightarrow v(t) = B_1 e^{-\alpha t} \cos w_d t + B_2 e^{-\alpha t} \sin w_d t$

$$w_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^{10} - 10^8} = 10^4 \sqrt{99} = 99498.7 \text{ rad/sec}$$

At $t = 0^+ \rightarrow$

$v(0^+) = V_0 = 0$ voltage across the capacitor is 0

Capacitor is a short therefore $I_C(0^+) = -I_L(0^+) = -I_0 = -10$ mA $\rightarrow dv(0^+)/dt = i_C(0^+)/C = -5 \times 10^5$

Apply the above initial conditions to $v(t)$ to find B_1 and B_2

$$v(0^+) = B_1 e^{-\alpha t} \Rightarrow B_1 = 0$$

$$\frac{dv(t)}{dt} = -B_1 \omega_d e^{-\alpha t} \sin \omega_d t + B_1 (-\alpha) e^{-\alpha t} \cos \omega_d t + B_2 \omega_d e^{-\alpha t} \cos \omega_d t + B_2 (-\alpha) e^{-\alpha t} \sin \omega_d t$$

$$\frac{dv(0^+)}{dt} = B_2 (-5 \times 10^5) = 99498.7 \Rightarrow B_2 = -0.2$$

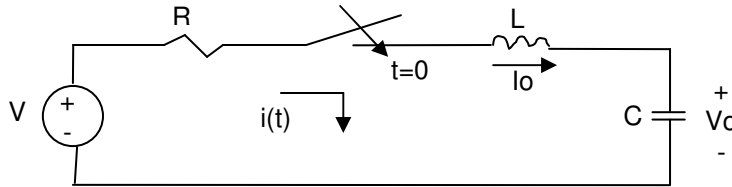
Therefore

$$v(t) = -0.2 e^{-10000t} \sin 99498.7t \text{ V for } t > 0$$

It is recommended that the reader plot the $i_L(t)$ to see an example of Underdamped signal.

8.4. Step Response for Series RLC Circuit

This section introduces the Series RLC circuit step response analysis. At time, $t=0$, the switch is closed which means current starts to flow to the RLC part of the circuit. For this derivation, it is assumed that energy stored in C as represented by v_o and the energy in L as represent by I_o are both equal to zero.



KVL \rightarrow

$$v_C + v_L + v_R = V$$

$$\frac{1}{C} \int_0^t i dt + I_0 + L \frac{di}{dt} + Ri = V$$

differentiate and re-arrange

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = 0$$

Comparing the above equation to the characteristic equation: $\frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = 0 \rightarrow$

$$\alpha = \frac{R}{2L} \text{ rad/sec} \quad \text{Neper Frequency}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ rad/sec} \quad \text{Resonant Radian Frequency}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \text{ rad/sec} \quad \text{Damped Radian Frequency}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 1}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 2}$$

Depending on the relationship between ω_0 and α , the response “output voltage, $v(t)$ ” may be Overdamped ($\omega_0^2 < \alpha^2$), Underdamped ($\omega_0^2 > \alpha^2$) or Critically damped ($\omega_0^2 = \alpha^2$), one of the following three types:

$$\text{Overdamped} (\omega_0^2 < \alpha^2) \Rightarrow i(t) = I_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\text{Critically Damped} (\omega_0^2 = \alpha^2) \Rightarrow i(t) = I_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$\text{Underdamped} (\omega_0^2 > \alpha^2) \Rightarrow i(t) = I_f + B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$\text{Overdamped}(\omega_0^2 < \alpha^2) \Rightarrow v_C(t) = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$\text{Critically Damped}(\omega_0^2 = \alpha^2) \Rightarrow v_C(t) = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

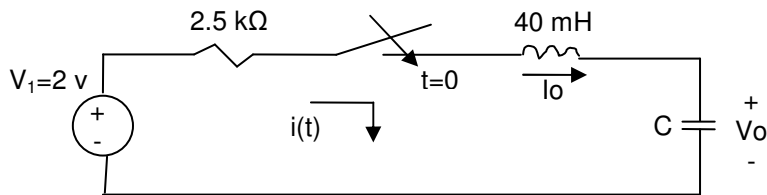
$$\text{Underdamped}(\omega_0^2 > \alpha^2) \Rightarrow v_C(t) = V_f + B_1' e^{-\alpha t} \cos w_d t + B_2' e^{-\alpha t} \sin w_d t$$

Where:

- $A_1, A_2, B_1, B_2, D_1, D_2, A_1', A_2', B_1', B_2', D_1', D_2'$ coefficients can be found by using the initial conditions:
 - For voltage use $v(0^+) = V_0$ and $dv(0^+)/dt$ which are derived from the capacitor voltage.
 - For current use $i(0^+) = I_0$ and $di(0^+)/dt$ which are derived from the inductor current.
- V_f is the value of voltage source or the capacitor voltage as t approaches infinity.
- I_f is the value of current source or the inductor current as t approaches infinity.

❖ Example – Series RLC Step Response

For the following circuit where the switch closes at time $t=0$ after being open for a long time with no stored energy:



- Find the minimum value C to the nearest μF so that this circuit is overdamped.
- Find the expression for $v_C(t)$

Solution

- Find the minimum value C to the nearest μF so that this circuit is overdamped.

$$\alpha = \frac{R}{2L} = \frac{2500}{2 \times 0.04} = 31250 \text{ rad/sec} \quad \text{Neper Frequency}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{25}{C}} \text{ rad/sec} \quad \text{Resonant Radian Frequency}$$

$$\text{Overdamped when } \omega_0^2 < \alpha^2$$

$$\frac{25}{C} < (31250)^2 \Rightarrow C > 2.56 \times 10^{-8} \Rightarrow C_{Min} = 1 \mu F$$

- Find the expression for $v_C(t)$

We know the circuit is overdamped therefore

$$\text{Overdamped}(\omega_0^2 < \alpha^2) \Rightarrow v_C(t) = V + A_1' e^{s_1 t} + A_2' e^{s_2 t} \text{ where } V=2V.$$

Since no energy is stored in the circuit both $V_C(0^+)$ and $dv_C(0^+)/dt = 0 \rightarrow$

$$v_C(0^+) = 2 + A_1' + A_2' = 0$$

$$dv_C(0^+)/dt = 2 + A_1' s_1 + A_2' s_2 = 0$$

Calculate Characteristic roots s_1 & s_2 :

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -31250 + \sqrt{31250^2 - \frac{25}{1 \times 10^{-6}}} = -402.6 \text{ rad / sec}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -31250 - \sqrt{31250^2 - \frac{25}{1 \times 10^{-6}}} = -62097.4 \text{ rad / sec}$$

Substitute and solve for A_1' and A_2' :

$$A_1' + A_2' = -2$$

$$-402.6A_1' - 62097.4A_2' = -2$$

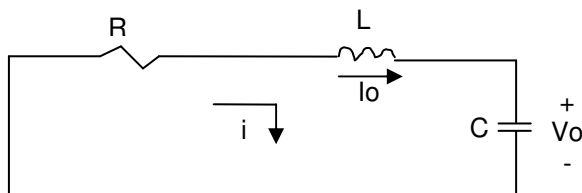
$$\rightarrow A_1' = -2.013 \text{ and } A_2' = 0.013$$

Therefore:

$$\text{Overdamped} (\omega_0^2 < \alpha^2) \Rightarrow v_C(t) = 2 - 2.013e^{-402.6t} + 0.013e^{-62097.4t}$$

8.5. Natural Response for Series RLC Circuit

This section introduces the series RLC circuit natural response analysis. At time, $t=0$, the switch is open which means current start to flow to the RLC part of the circuit. For this derivation, it is assumed that energy stored in C as represented by v_o and the energy in L as represent by I_o .



$$v_C + v_L + v_R = 0$$

$$L \frac{di}{dt} + \frac{1}{C} \int_0^t i dt + V_0 + Ri = 0$$

differentiate and re-arrange

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = 0$$

Comparing the above equation to the characteristic equation: $\frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = 0 \rightarrow$

$$\alpha = \frac{R}{2L} \text{ rad/sec} \quad \text{Neper Frequency}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ rad/sec} \quad \text{Resonant Radian Frequency}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \text{ rad/sec} \quad \text{Damped Radian Frequency}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 1}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \text{ rad/sec} \quad \text{Characteristic Root or Complex Frequency 2}$$

Depending on the relationship between ω_0 and α , the response “output voltage, $v(t)$ ” may be Overdamped ($\omega_0^2 < \alpha^2$), Underdamped ($\omega_0^2 > \alpha^2$) or Critically damped ($\omega_0^2 = \alpha^2$), one of the following three types:

$$\text{Overdamped}(\omega_0^2 < \alpha^2) \Rightarrow i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\text{Critically Damped}(\omega_0^2 = \alpha^2) \Rightarrow i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$\text{Underdamped}(\omega_0^2 > \alpha^2) \Rightarrow i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$\text{Overdamped}(\omega_0^2 < \alpha^2) \Rightarrow v_C(t) = A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$\text{Critically Damped}(\omega_0^2 = \alpha^2) \Rightarrow v_C(t) = D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

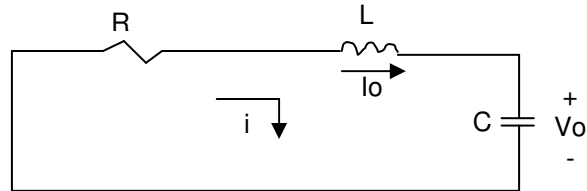
$$\text{Underdamped}(\omega_0^2 > \alpha^2) \Rightarrow v_C(t) = B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t$$

Where:

- $A_1, A_2, B_1, B_2, D_1, D_2, A_1', A_2', B_1', B_2', D_1', D_2'$ coefficients can be found by using the initial conditions:
 - For voltage use $v(0^+) = V_0$ and $dv(0^+)/dt$ which are derived from the capacitor voltage.
 - For current use $i(0^+) = I_0$ and $di(0^+)/dt$ which are derived from the inductor current.

❖ Example – Series RLC Natural Response

For the following circuit the initial current through inductor is 10 mA and initial voltage through the capacitor is 5 volts at time $t=0$.



If $R=40 \text{ k}\Omega$ and $C=2 \text{ nF}$, what value of L ensures that the circuit is critically damped.

Solutions

$$\alpha = \frac{R}{2L} = \frac{20000}{L} \text{ rad/sec}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{5 \times 10^8}{L}} \text{ rad/sec}$$

For critically damped RLC circuits $\rightarrow \omega_0^2 = \alpha^2 \rightarrow$

$$\left(\frac{20000}{L}\right)^2 = \frac{5 \times 10^8}{L}$$

$$L = \frac{4}{5} = 0.8 \text{ H}$$

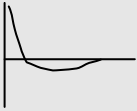
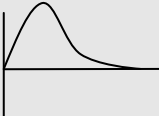

8.6. Summary

This section summarizes all four type of RLC circuit Responses into a general form where $x(t)$ may be current or voltage signal. To use the equation here, simply apply the appropriate initial conditions to find the constant and circuit parameters.

The general equations describing the circuit and its behavior:

- Characteristic equation: $\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + w_0^2x = 0$
- Complex Roots (rad/sec.): $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -\alpha \pm jw_d$
- Neper Frequency (rad/sec.):
 - $\alpha = \frac{1}{2RC}$ for Parallel RLC
 - $\alpha = \frac{R}{2L}$ for Series RLC
- Resonant Radian Frequency rad/sec.): $w_0 = \frac{1}{\sqrt{LC}}$
- Damped Radian Frequency (rad/sec.): $w_d = \sqrt{w_0^2 - \alpha^2}$

The following table summarizes the responses of all RLC circuits discussed in this chapter:

Overdamped $w_0^2 < \alpha^2$ $x(t)$ approaches final value without Oscillation. 	Critically Damped $w_0^2 = \alpha^2$ $x(t)$ is on the verge of Oscillating about its final value. 	Underdamped $w_0^2 > \alpha^2$ $x(t)$ oscillates about its final value. 
Natural Response		
$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ <i>Coef.Cal.</i> $x(0) = A_1 + A_2$ $\frac{dx(0)}{dt} = A_1 s_1 + A_2 s_2$	$x(t) = (D_1 t + D_2) e^{-\alpha t}$ <i>Coef.Cal.</i> $x(0) = D_2$ $\frac{dx(0)}{dt} = D_1 - \alpha D_2$	$x(t) = (B_1 \cos w_d t + B_2 \sin w_d t) e^{-\alpha t}$ <i>Coef.Cal.</i> $x(0) = B_1$ $\frac{dx(0)}{dt} = -\alpha B_1 + w_d B_2$
Step Response (X_f is Source Value)		
$x(t) = X_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$ <i>Coef.Cal.</i> $x(0) = X_f + A_1' + A_2'$ $\frac{dx(0)}{dt} = A_1' s_1 + A_2' s_2$	$x(t) = X_f + (D_1' t + D_2') e^{-\alpha t}$ <i>Coef.Cal.</i> $x(0) = X_f + D_2'$ $\frac{dx(0)}{dt} = D_1' - \alpha D_2'$	$x(t) = X_f + (B_1' \cos w_d t + B_2' \sin w_d t) e^{-\alpha t}$ <i>Coef.Cal.</i> $x(0) = X_f + B_1'$ $\frac{dx(0)}{dt} = -\alpha B_1' + w_d B_2'$

8.7. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 8.

8.8. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 9. Sinusoidal Steady State Analysis

Key Concepts and Overview

- ❖ Sinusoidal Source & Steady State Response
- ❖ Root Mean Square (RMS) Value and Total Response
- ❖ Phasor Domain Definition
- ❖ Passive Circuit Elements in Phasor Domain
- ❖ Phasor Domain Circuit Analysis Techniques
- ❖ Phasor Domain Circuit Analysis Exercises
- ❖ Additional Resources

9.1. Sinusoidal Source & Steady State Response

In the last two chapters, circuits with R, L and C were introduced and analyzed for their performance during turn on phase (Step Response) and turn off phase (Natural Response). This chapter focuses on the steady state operation phase. The steady state phase starts after turn on phase has stabilized and before impact of turn off phase has affected the circuit operation.

Further in this section, all sources are assumed to be sine or cosine functions which are generally referred to as the sinusoidal sources and general forms of RLC circuits are analyzed in their Steady State. The output under these conditions is referred to as the Sinusoidal Steady State Response.

Sinusoidal source can be expressed either as a cosine or as a sine since $\cos(x) = \sin(x - 90^\circ)$. In Electrical Circuit analysis, the general Sinusoidal source is typically represented by Cosine as shown below:

$$v(t) = V_m \cos(\omega t + \phi) \text{ for Sinusoidal Voltage Source}$$

or

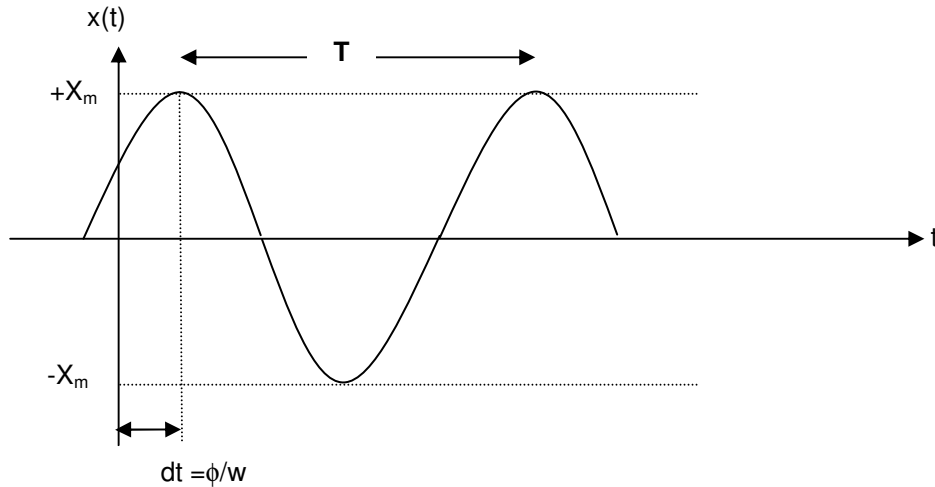
$$i(t) = I_m \cos(\omega t + \phi) \text{ for Sinusoidal Current Source}$$

where:

- ω = angular frequency in radian/second
 $\omega = 2\pi/T$ where $T = 1/f$ period in seconds
 $\omega = 2\pi f$ where $f = 1/T$ is the frequency in Hertz(cycles/second)
"degree = radian * $180/\pi$ "
- t is time in seconds.
- I_m and V_m are maximum amplitude of current/voltage also called peak value.
- ϕ is the phase shift which is basically a measure of dt , the distance from $t=0$ of the start of the period.
 - Meaning of sign of ϕ
 - (1) If $\phi > 0$ the signal is shifted to the left
 - (2) If $\phi < 0$ the signal is shifted to the right
 - Relationship between ϕ & dt
 $\phi = (dt * \omega)$ radians which is the phase shift in radian or $dt = \phi/\omega$ which is shift in time in seconds.

It is common to express " ω " in radians/sec and " ϕ " in radians since the process requires less conversions than if degrees were used instead of radians.

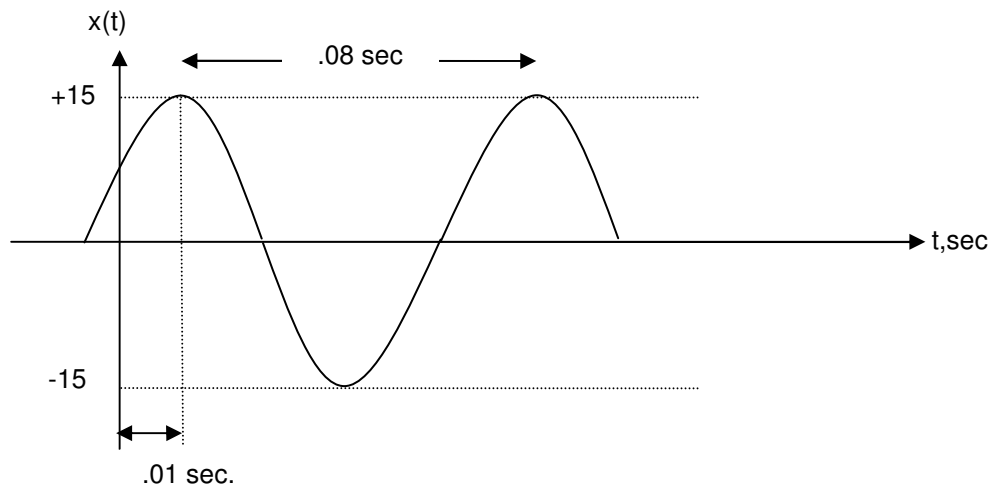
Below is a diagram of typical generalized sinusoidal signal, $x(t) = X_m \cos(\omega t - \phi)$:



A Sinusoidal signal is completely defined when its frequency, maximum amplitude and phase angle is given or can be derived from the available information.

➤ Example – Sinusoidal Signal

Write the Sinusoidal signal equation for the Following $x(t)$ function:



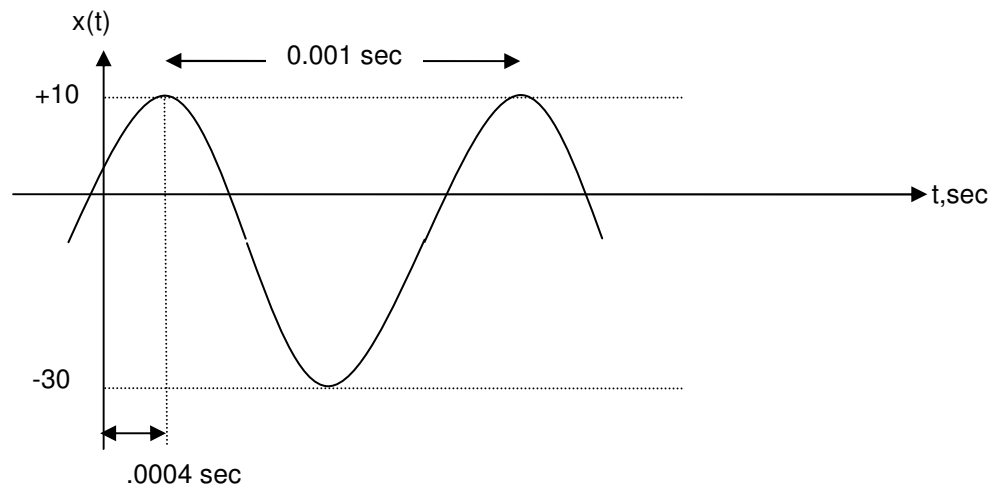
Solutions:

$$\begin{aligned} \text{Phase shift} &= dt * (2\pi/T) = 0.01 * (2\pi/0.08) = \pi/4 \text{ rad} \\ \text{DC offset} &= (\text{Max} + \text{Min}) / 2 = 0 \text{ V} \\ \text{Radial Freq} = \omega &= 2\pi f = 2\pi/T = 2\pi/0.08 = 25\pi \text{ rad/sec} \end{aligned}$$

$$x(t) = 15 \cos(25\pi t - \pi/4) \text{ V}$$

➤ Example – Sinusoidal Signal

Write the Sinusoidal signal equation for the Following $x(t)$ function:



Solutions:

9.2. Root Mean Square (RMS) Value and Total Response

Before focusing on steady state response analysis, it is important to consider two other type of analysis for circuit with sinusoidal signals (RMS and Total Responses).

Root Mean Square (RMS) Value is such named because it is calculated by taking the square root of averages of the squared signal value over a single period as shown below.

$$V_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) dt} = \frac{V_m}{\sqrt{2}} = \frac{\sqrt{2} V_m}{2} \text{ for Sinusoidal voltage only}$$

$$I_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} I_m^2 \cos^2(\omega t + \phi) dt} = \frac{I_m}{\sqrt{2}} = \frac{\sqrt{2} I_m}{2} \text{ for Sinusoidal current only}$$

RMS value is useful in understanding the voltage, current and power equivalent of the sinusoidal signal as if it was a constant DC voltage and current. It is commonly used to calculate average power delivered by the sources or absorbed by an element.

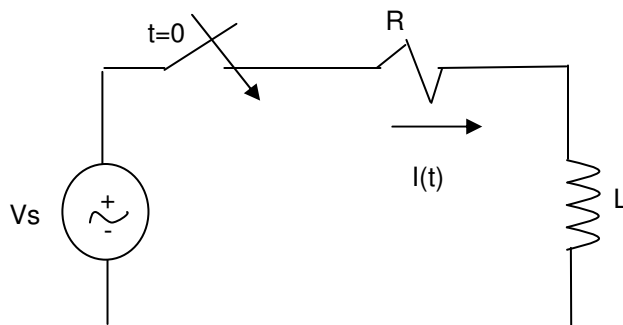
The following trigonometry equalities are used to prove that RMS value of a sinusoidal signal is

$$X_{rms} = \frac{X_m}{\sqrt{2}} :$$

- $\sin^2 x + \cos^2 x = 1$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(2x) = \cos^2 x - \sin^2 x$
- $\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$

Another concept to consider is the Total Response. Total Response includes a transient portion and a steady state portion. As the name implies transient portion goes to zero after the circuit has been in operation for some time. On the other hand the steady state portion will continue to present until the source has been removed.

An example is the best way to demonstrate the concept. For the following circuit find the total response, $i(t)$ for $t > 0$.



Given:

- * Initial State Energy is 0 at $t=0$
- * $V_s = V_m \cos(\omega t + \phi)$

$$L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

Solving the above DFQ

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta) \quad \text{Where } \theta = \tan^{-1}(\omega L / R)$$

The total response includes:

- Transient response (first term) which reduces in magnitude over time

$$\text{Transient response of } i(t) = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}$$

- Steady State Response which is periodic therefore will not diminish over time

$$\text{Steady State Response } i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

where:

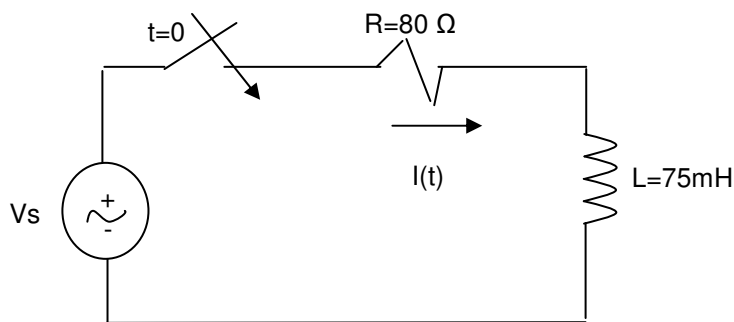
(1) Output signal Frequency is the same as input signal frequency (ω).

(2) Maximum amplitude of response is $\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$ when input maximum amplitude is V_m .

(3) The phase angle of the output will generally be different from the input signal unless the circuit is purely resistive (no inductors or capacitors)

❖ Example – Total response

The Voltage applied to the circuit shown below at $t=0$ is $20\cos(800t + 25^\circ)$ where the Initial current is 0 at $t=0$.



Find the Total Response $i(t)$ and identify its transient and steady state response:

Solution

$$V_s = V_m \cos(\omega t + \phi) = 20 \cos(800t + 25^\circ) \Rightarrow V_m = 20, \quad \omega = 800, \quad \phi = 25^\circ$$

$$\theta = \tan^{-1}(\omega L / R) = \tan^{-1}(800 * 0.075 / 80) = 36.9^\circ$$

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

$$i = (-0.2)(\cos(25 - 36.9))e^{-1066.67t} + (0.2)\cos(800t + 25 - 36.9)$$

$$i = .196e^{-1066.67t} + 0.2\cos(800t - 11.9^\circ)$$

Therefore:

$$\text{Transient Response for } i(t) = 0.196e^{-1066.67t}$$

$$\text{Steady State Response for } i(t) = 0.2\cos(800t - 11.9^\circ)$$

As we discussed earlier, the resulting differential equation may be difficult or not possible to solve directly in order to obtain the total response. Since the transient response is only of interest during the initial start up phase, it will not be needed in steady state analysis which is the most common circuit analysis .

In the following section, Phasor or Frequency Domain analysis will be introduced for calculating the steady state response of arbitrary RLC circuits without needing to use differential equations. Phasor Domain analysis enables the use of algebra to find the RLC steady state circuit response.

9.3. Phasor Domain Definition

All the previous analysis has been in time domain where the independent variable is time. In this section, Frequency or Phasor Domain will be introducing where time is not part of the equation. But before starting the circuit analysis in Phasor domain, it is important to introduce and review a number of key concepts.

Start with some basic trigonometry equalities which are commonly used in analysis:

- Euler's identity is foundation of the transformation.

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \quad \text{or} \quad \cos\theta = \frac{e^{+j\theta} + e^{-j\theta}}{2} \quad \text{or} \quad \sin\theta = \frac{e^{+j\theta} - e^{-j\theta}}{2j}$$

- Real and Imaginary part of complex number ($e^{j\theta}$)

$$\text{Real Part} = \Re(e^{j\theta}) = \cos\theta$$

$$\text{Imaginary Part} = \Im(e^{j\theta}) = \sin\theta$$

With the above information, It is time to introduce Phasor transformation. By applying Euler's identity, the sinusoidal signal may be rewritten as shown below:

$$v(t) = V_m \cos(\omega t + \phi) = V_m \Re\{e^{j(\omega t + \phi)}\} = V_m \Re\{e^{j\omega t} e^{j\phi}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}$$

In this chapter, only linear RLC circuits are analyzed which means the circuit frequency is the same for all input and output. So in the Phasor Domain, there is no need to carry the $e^{j\omega t}$ (it is implied) therefore Phasor representation of the sinusoidal signal may be written as $V_m e^{j\phi}$. In other words, the Phasor representation of $V_m \cos(\omega t + \phi)$ carries the phase angle and amplitude information. In other words, the frequency is assumed to be " ω " and is not explicitly shown in the Phasor representation of the signal.

$P[x(t)]$ is the notation used to indicate Phasor transformation of $x(t)$ which is shown below:

$$\text{Phasor Transform or Phasor Representation} = P[V_m \cos(\omega t + \phi)] = V_m e^{j\phi}$$

This transformation transfers the sinusoidal function from the time domain to the complex number domain which is also called the Frequency or Phasor Domain since now the response mainly depends on phase.

In Phasor domain, Phasor variables are shown as capital letter and may be presented in one of the following three accepted Phasor representations:

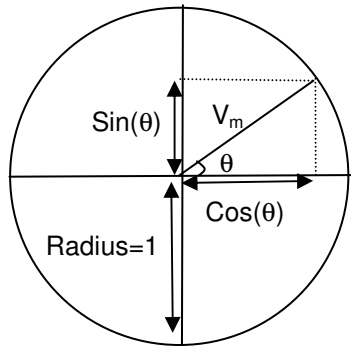
- Polar Form:** $V = V_m e^{\pm j\phi}$
- Rectangular Form:** $V = V_m \cos\phi \pm jV_m \sin\phi$

Note: conversion between Polar and rectangular form uses $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

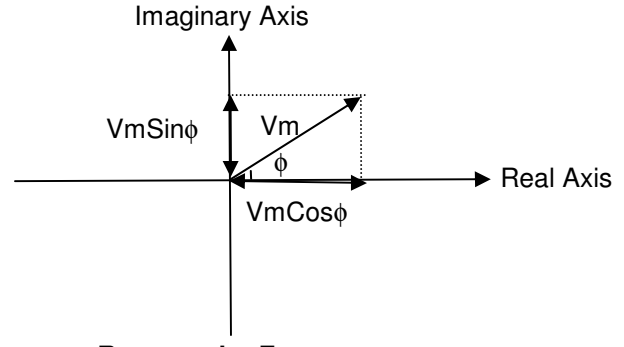
- Angular Form:** $V = V_m \angle \phi^\circ$

Note: Angular Form is a variation of Polar Form.

In all three formats, the maximum amplitude (Peak value) and phase are present. Of course the frequency is the same as ω (implicit in the Phasor Transformation). It may be useful to revisit Unit Circle and Complex Coordinate systems in order to show the relationship between the three Phasor forms:



Polar & Angular Form on Unit Circle



Rectangular Form on Complex Rectangular Coordinate System

So far, transformation from Time to Phasor domain has been shown. The inverse transformation may be represented by the following notation and relationship:

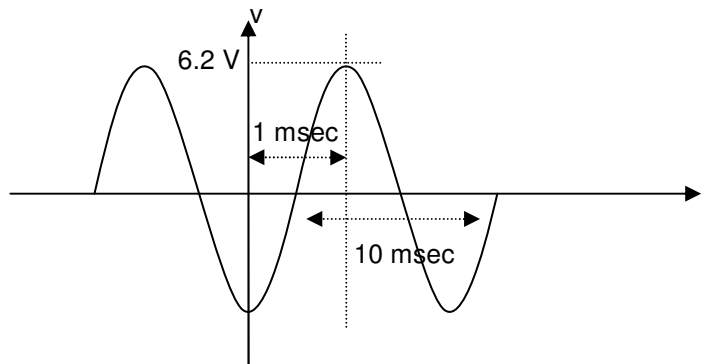
$$P^{-1} \{V_m e^{j\phi}\} = \Re \{V_m e^{j\phi} e^{j\omega t}\} = V_m \cos(\omega t + \phi)$$

The benefit of Phasor domain is that it simplifies analysis of circuit in steady state mode with sinusoidal input. This simplification is the result of the fact that Phasor domain only requires solutions to algebraic equations instead of the more complex differential equations.

Proper use of Phasor and Time domain dictate that analysis must be completed either fully in Time domain or fully in Phasor domain. It is incorrect to mix the two on the same circuit analysis. All the analysis techniques such as Kirchhoff's Laws and associated methods apply to the Phasor domain in much the same way as they did to the Time domain analysis.

❖ **Example - Phasor transformation**

Find time-based equation and Phasor transformation for the Voltage $v(t)$ signal:



Solution:

From the above diagram, it is observed that:

Maximum voltage, $V_m = 6.2 \text{ V}$

Period, $T = 10 \text{ msec} \rightarrow$ Linear frequency, $f = 1/T = 100 \text{ Hz} \rightarrow$

Radial Frequency, $\omega = 2\pi f = 2\pi/T = 200\pi \text{ radians/Sec.}$

$dt = -1 \text{ msec} \rightarrow \phi = \omega \cdot dt = 200\pi \cdot (-10^{-3}) = -\pi/5$

Therefore the Time domain signal may be presented as:

$$v(t) = V_m \cos(\omega t + \phi) = 6.2 \cos(200\pi t - \pi/5) = 6.2 \cos(200\pi t - \pi/5)$$

Phasor Representations are:

$$\text{Polar Form: } V = V_m e^{+j\phi} = 6.2 e^{-j(\frac{\pi}{5})}$$

$$\text{Rectangular Form: } V = V_m \cos\phi \pm jV_m \sin\phi = 6.2 \cos(\pi/5) - j6.2 \sin(-\pi/5)$$

$$\text{Angular Form: } V = V_m \angle \phi = 6.2 \angle -\pi/5$$

9.4. Passive Circuit Elements in Phasor Domain

In the previous section, the transformation of signals (current and voltage sources) between time and Phasor domain was discussed. In this section, the focus is on conversion of elements (R, L and C) value between Time domain and Phasor domain. The transformation of passive elements is based on the relationship between Phasor Voltage (V) & Phasor Current (I). The general term for V/I is impedance and starting with section, impedance will be used to refer to the value V/I . You may recall for a purely resistive circuit, we referred to $V/I=R$ as resistance which is a special case of impedance.

In this section, Phasor domain element values for R, L and C will be derived along with their relationship to the corresponding Time domain element values.

❖ Resistor's Impedence in Phasor domain ($Z=R$)



Resistor's current and voltage in the Time domain can be driven as shown below:

$$\text{For the given } i(t) = I_m \cos(\omega t + \theta_i) \rightarrow$$

$$v(t) = R I_m \cos(\omega t + \theta_i)$$

$$\text{Therefore } R = v(t)/i(t)$$

Now, transform the current and voltage equation to Phasor Domain as shown below:

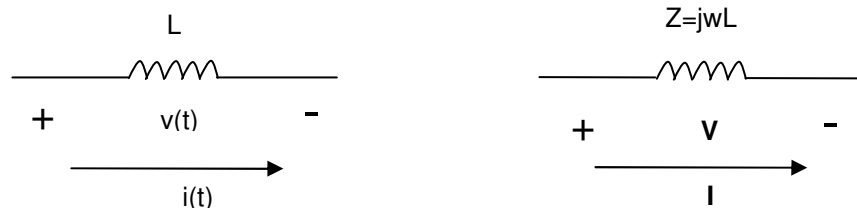
$$I = I_m e^{\pm j\phi_i}$$

$$V = R I_m e^{\pm j\phi_i}$$

Therefore $Z = V/I = R$ "Impedance for resistor in Phasor Domain"

- Observations:
 - (1) I & V are in-phase (no phase shift)
 - (2) Resistor only impacts the signal magnitude

❖ Inductor's Impedence in Phasor domain ($Z=j\omega L$)



Inductor's Current and voltage in the Time domain is derived below:

$$\text{For the given } i(t) = I_m \cos(\omega t + \theta_i) \rightarrow$$

$$v(t) = L di/dt = -\omega L I_m \sin(\omega t + \theta_i)$$

Using the fact that sine and cosine are only shifted by 90° , $v(t)$ can be written as:

$$v(t) = -\omega L I_m \cos(\omega t + \theta_i - 90^\circ)$$

Now, writing the Phasor transformation of $i(t)$ and $v(t)$ results in:

$$\text{Phasor Transformation of } v(t) \Rightarrow V = -\omega L I_m e^{j(\phi_i - 90^\circ)} = -\omega L I_m e^{j\phi_i} e^{-j90^\circ}$$

$$\text{Apply Euler's Identity } e^{-j90^\circ} = \cos 90^\circ - j \sin 90^\circ = -j \Rightarrow V = j\omega L I_m e^{j\phi_i}$$

$$\text{Phasor Transformation of } i(t) \Rightarrow I = I_m e^{j\phi_i}$$

Therefore

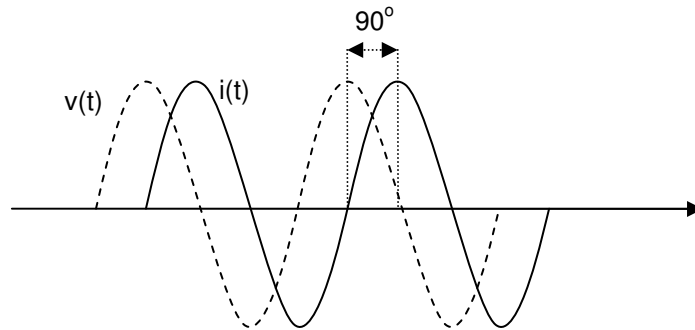
$$Z = \frac{V}{I} = j\omega L \quad \text{"Impedance of inductors in Phasor Domain"}$$

- Observations:

- (1) Current flowing through inductor will result in voltage with 90 degree phase lead (+90°)

$$V = \omega L I_m \angle (\theta_i + 90^\circ)$$

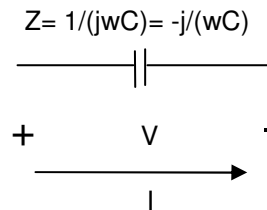
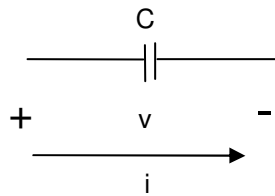
* Voltage leads current or in other words current lags voltage in Inductors as shown below:



- (2) L impacts both the amplitude and phase

It has shown the impact on the phase (90 degrees shift) and the amplitude $V_m = \omega L I_m$ as shown by earlier equation $V = \omega L I_m \angle (\theta_i + 90^\circ)$.

❖ Capacitor's impedance in Phasor domain ($Z = 1/(j\omega C) = -j/\omega C$)



Capacitor's current and voltage in the Time domain can be derived as shown below:

$$\text{For the given } v(t) = V_m \cos(\omega t + \theta_v) \rightarrow$$

$$i(t) = C \frac{dv(t)}{dt} = -\omega C V_m \sin(\omega t + \theta_v)$$

Using the fact the sine and cosine are only shifted by 90°, $i(t)$ can be rewritten as:

$$i(t) = -\omega C V_m \cos(\omega t + \theta_v - 90^\circ)$$

Now, writing the Phasor transformation of $i(t)$ and $v(t)$ results in:

$$\text{Phasor Transformation of } i(t) \Rightarrow I = -\omega C V_m e^{j(\phi_j - 90)} = -\omega C V_m e^{j\phi_j} e^{-j90}$$

$$\text{Apply Euler's Identity } e^{-j90} = \cos 90 - j \sin 90 = -j \Rightarrow V = j\omega C V_m e^{j\phi_j}$$

$$\text{Phasor Transformation of } v(t) \Rightarrow V = V_m e^{j\phi_j}$$

Therefore

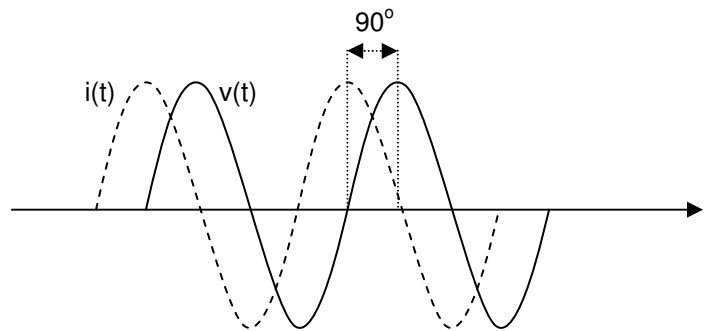
$$Z = \frac{V}{I} = \frac{1}{j\omega C} = -\frac{j}{\omega C} \quad \text{"Impedance of capacitors in Phasor Domain"}$$

- Observations:

(1) Voltage across capacitor results in current with 90 degree phase lead (+90°)

$$I = \omega C V_m \angle (\theta_v + 90)^\circ$$

* Current leads voltage or in other words voltage lags current in capacitors



(2) C impacts both the amplitude and phase

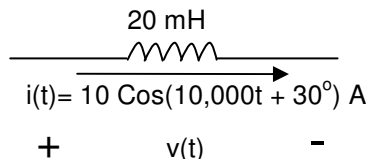
It has been shown the impact on the phase (90 degrees) and the amplitude $I_m = \omega C V_m$ as shown by earlier equation $I = \omega C V_m \angle (\theta_v + 90)^\circ$.

❖ Summary of Relationships and additional terminology

Element	Impedance ($Z=V/I$) In Ohms	Reactance (Imaginary Part of Z)	Admittance ($Y=I/V=1/Z$) in Mhos	Susceptance (Imaginary part of Y)
Resistor	R (resistance)	0	$G=1/R$ (conductance)	0
Capacitor	$1/j\omega C = -j/\omega C$	$-1/\omega C$	$j\omega C$	ωC
Inductor	$j\omega L$	ωL	$1/j\omega L = -j/\omega L$	$-1/\omega L$

- Impedance may be a complex number ($a + jb$)
- $j*j = -1$ so $1/j = -j$ (multiply top & bottom by j)
- R , L and C use passive sign convention which means current flows through the element in the direction of voltage drop (from + to – voltage).

- ❖ Example – Phase and Time Domain Transformations for a Steady State Circuit
For the following circuit element:



- Find the inductive reactance and the impedance of the inductor
- Find the Steady State Time domain voltage expression
- Find the Steady State Phasor domain voltage expression

Solution

- Find the inductive reactance and the impedance of the inductor
Compare $i = 10 \cos(10,000t + 30^\circ)$ mA with standard for $i = I_m \cos(\omega t + \Phi) \rightarrow \omega = 10,000$
Therefore Reactance $= \omega L = (10000)(20 \times 10^{-3}) = 200 \Omega$ and
Impedance $= j\omega L = j200 \Omega$

- Find the Steady State Time domain voltage expression
 $V = (j\omega L)I = j2000 \cos(10,000t + 30^\circ) \text{ V}$

- the Phasor Voltage V

Note : $J = J \sin 90^\circ + J \cos 90^\circ = e^{j90}$

$$V = 2e^{j90}e^{j30} = 2e^{j120} \text{ kV} \quad \text{"Polar Form"}$$

$$V = 2 \angle 120^\circ \text{ kV} \quad \text{"Angular Form"}$$

$$V = 2 \cos(120^\circ) + j2 \sin(120^\circ) = 2 + j1.7 \text{ kV} \quad \text{"Rectangular Form"}$$

9.5. Phasor Domain Circuit Analysis Techniques

In the earlier sections of this chapter, Phasor transformation of signals and elements values have been described. Once the circuit (elements and sources) has been transformed into the Phasor domain, all the circuit analysis techniques developed for circuit analysis applies where R is replaced by Z. The only difference being that R was always real number where Z may be a complex number.

By replacing the R with Z, all the circuit analysis tools learned in Previous sections which are listed below apply:

- Kirchhoff's current and voltage law
- Node Voltage and Mesh Current method
- Parallel and Series simplification
- Source Transformation
- Thevenin and Norton equivalent
- ...

For example Kirchhoff's Voltage (KVL) in Phasor domain require writing " $V_1 + V_2 + V_3 + \dots + V_n = 0$ " when these are Phasor representation of voltages around a mesh. When using Kirchhoff's Current Law (KCL) in Phasor domain, the equation " $I_1 + I_2 + I_3 + \dots + I_n = 0$ " applies where these are Phasor representation of currents out of a node. Other techniques such as Norton and Thevenin similarly apply.

Here are few pointers when applying Phasor analysis:

- Transform all the signals and elements in the circuit to Phasor domain and redraw the circuit in Phasor domain prior to start of your Phasor domain analysis.
- Typically, Rectangular Form ($a + jb$) of Phasor representation is best suited for equations requiring addition or subtraction to solve. While the Polar Form (de^{jk}) of Phasor representation works well for equations requiring multiplication and division to solve.
- Once the problem is solved then results may be transformed back to Time domain if needed.

Simplifications techniques such as Serial, Parallel and Delta-to-Wye Transformations may be used in Phasor analysis. As shown below, these techniques work the same as they did when the circuit only contained R instead of impedance, Z.

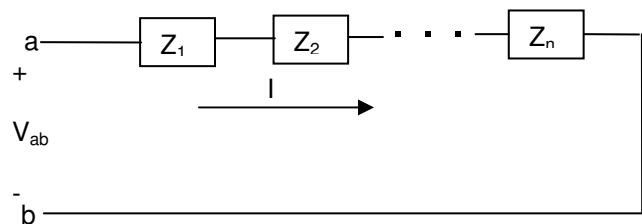
❖ Combining Impedances (Z) in Series

$$V_{ab} = Z_1 I + Z_2 I + \dots + Z_n I$$

$$Z_{ab} I = Z_1 I + Z_2 I + \dots + Z_n I$$

Divide by I

$$Z_{ab} = Z_1 + Z_2 + \dots + Z_n$$



❖ Combining Impedances (Z) in Parallel

$$I = I_1 + I_2 + \dots + I_n$$

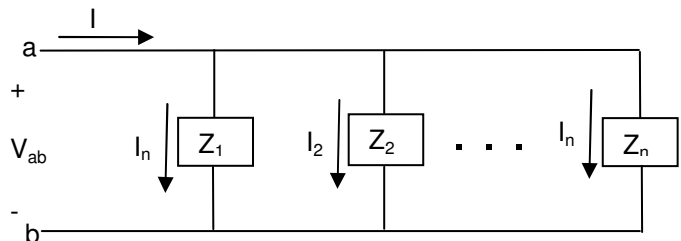
$$V_{ab} / Z_{ab} = V_{ab} / Z_1 + V_{ab} / Z_2 + \dots + V_{ab} / Z_n$$

Divide by V_{ab}

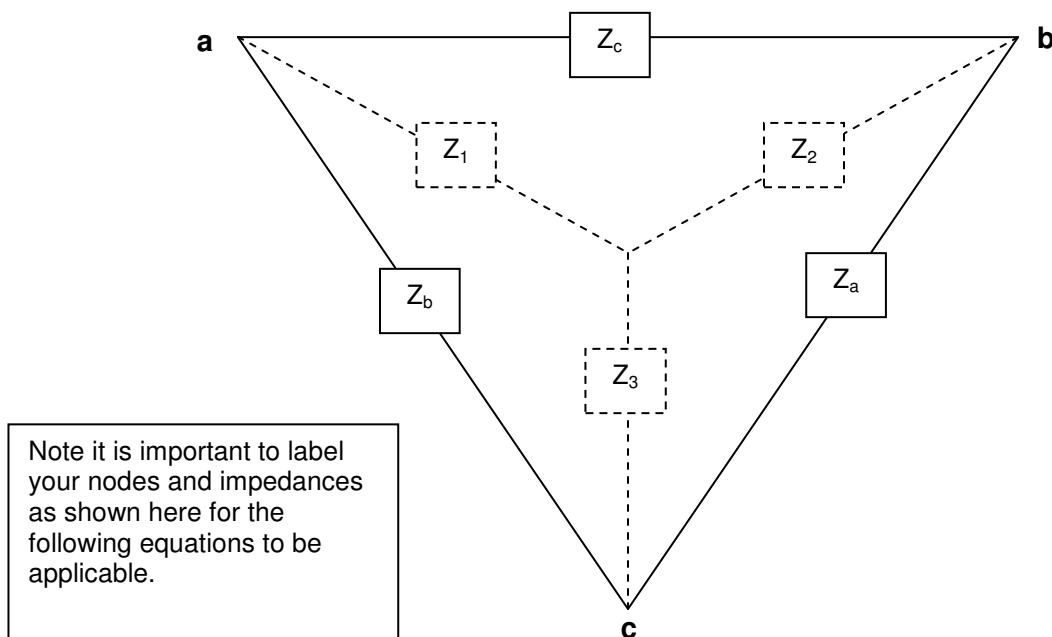
$$1 / Z_{ab} = 1 / Z_1 + 1 / Z_2 + \dots + 1 / Z_n$$

or

$$Y_{ab} = Y_1 + Y_2 + \dots + Y_n$$



❖ Simplification using Delta-to-Wye and Wye-to-Delta transformations



➤ Equations for transforming Delta-to-Wye

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

➤ Equations for transforming Wye-to-Delta

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_1}$$

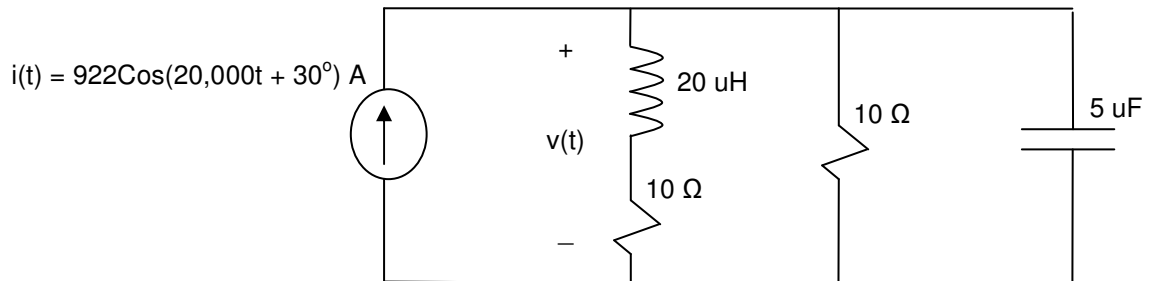
$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3}$$

9.6. Phasor Domain Circuit Analysis Exercises

This section provides a number of examples for the application of circuit analysis techniques from earlier chapters to the Phasor domain circuit analysis. As discussed in the previous section, these are techniques introduced for resistive circuit analysis which apply to RCL circuits when R (resistance, real number) is generalized to Z (impedance, complex number).

- ❖ Example – Application of KCL and Network Simplification
For the following circuit:



Find the expression for $v(t)$..

Solution

Here is step by step process of using Phasor Domain for the analysis:

Form the equation of the $i(t) = I_m \cos(\omega t + \Phi) = 922 \cos(20,000t + 30^\circ) \rightarrow$

Radial Frequency, $\omega = 20,000$ radians/sec

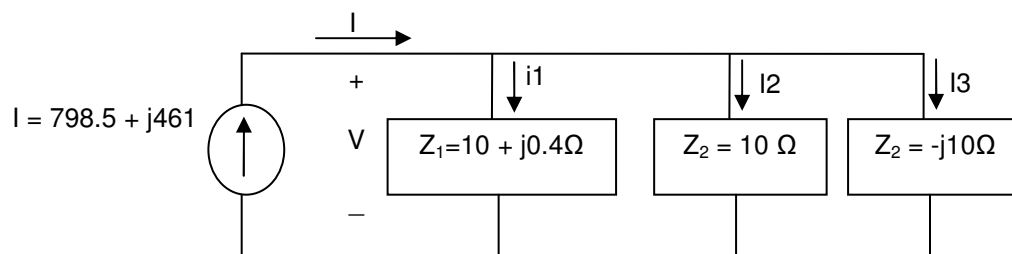
Maximum Current, $I_m = 922$ A

Phase, $\Phi = 30^\circ$

Step 1) Transform the circuit signals and elements to Phasor domain (user Rectangular form)

- Current $i(t)$ in rectangular form, $I = 922 \cos(30^\circ) + j922 \sin(30^\circ) = 798.5 + j461$ A
- Inductor, $20 \mu\text{H} \rightarrow Z = j\omega L = j(20000)(20 \times 10^{-6}) = j0.4 \Omega$
- Capacitor, $5 \mu\text{F} \rightarrow Z = 1/j\omega C = 1/j(20000)(5 \times 10^{-6}) = -j10 \Omega$
- Resistor value, R and Z , are the same in both domains

Step 2) Redraw the circuit in Phasor domain



Step 3 - Option A) Apply Kirchhoff's current law to solve find V

$$I = I_1 + I_2 + I_3 \rightarrow 798.5 + j461 = V/(10 + j0.4) + V/10 + V/-j10$$

Solve for V

Note that we can apply the $(a + bj)(a - bj) = a^2 + b^2$ to first fraction to have all imaginary part in the numerator:

$$798.5 + j461 = (10 - j0.4)V/(100 + j16) + V/10 + jV/10$$

$$798.5 + j461 = 0.1V - j0.004V + 0.1V + j0.1V$$

$$798.5 + j461 = 0.2V + j0.096V$$

$$V = (798.5 + j461) / (0.2 + j0.096)$$

Since there is a division, it may be easier to use angular or polar form to find the value of V.

$$V = \frac{922 \angle 30^\circ}{\sqrt{0.2^2 + 0.096^2} \angle \tan^{-1}\left(\frac{0.096}{0.2}\right)} = \frac{922 \angle 30^\circ}{.22 \angle -25.6^\circ} = 4190.9 \angle 30 - (-25.6)$$

$$V = 4190.9 \angle 55.6^\circ = V_m \angle \phi$$

Step 3 - Option B) Combine Impedances using parallel simplification

$$Z_{eq} = 1/(1/Z_1 + 1/Z_2 + 1/Z_3) = 1/(1/(10 + j0.4) + 1/10 + 1/-j10) = 4.07 + j1.95$$

$$\text{Then } V = I * Z_{eq} = (798.5 + j461)(4.07 + j1.95) = 2350.9 + j3433.3$$

$$V = 4190.9 \angle 55.6^\circ$$

Step 4) Transform Phasor domain form " $V = V_m \angle \Phi$ " to Time domain " $v(t) = V_m \cos(\omega t + \Phi)$ "

From derivation so far it is given that

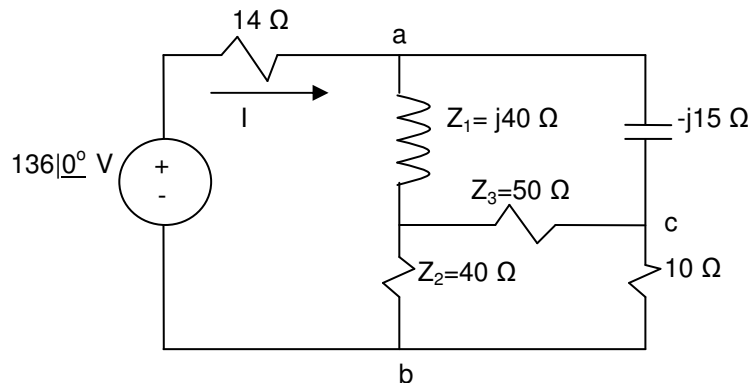
radial frequency, $\omega = 20,000$ radian/sec

peak voltage, $V_m = 4190.91V$

phase shift, $\Phi = 55.6^\circ$

$$\text{Therefore } v(t) = 4190.91 \cos(20,000t + 55.6^\circ)$$

- ❖ Example—Application of delta-to-Wye transformation
Find current I in the following circuit.



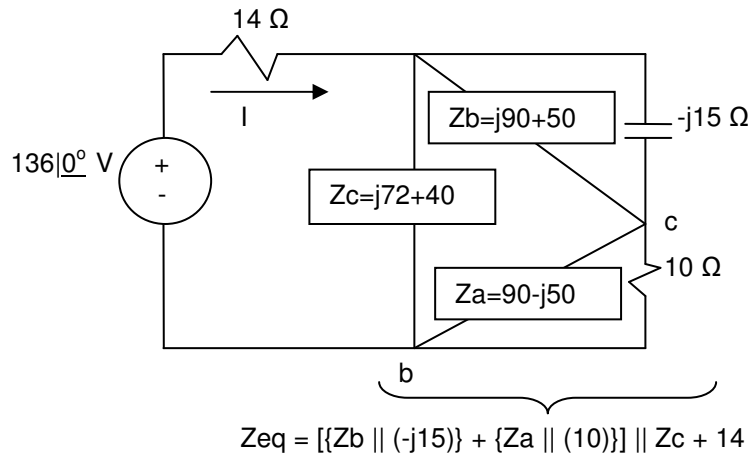
1) Apply Wye-to-Delta Transformation

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_1} = \frac{(j40 * 40 + 40 * 50 + j40 * 50)}{j40} = \frac{j3600 + 2000}{j40} = 90 - j50$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_2} = \frac{j3600 + 2000}{40} = j90 + 50$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3} = \frac{j3600 + 2000}{50} = j72 + 40$$

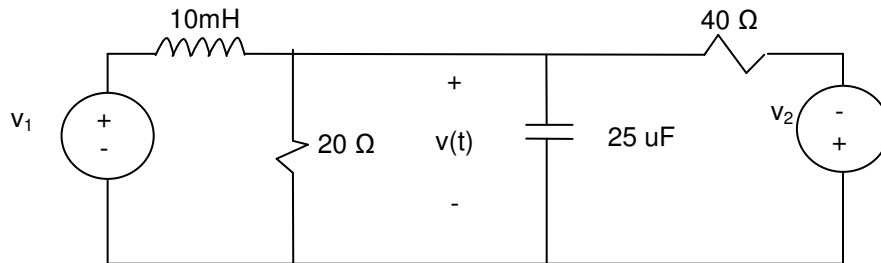
2) Redraw the circuit



$$Z_{eq} = \frac{1}{\frac{1}{\frac{1}{\left(\frac{1}{j90+50} + \frac{1}{-j15}\right)} + \frac{1}{\left(\frac{1}{90-j50} + \frac{1}{10}\right)}} + \frac{1}{j72+40}} + 14 = 30 - j15.9 = 34 \angle -28.07^\circ$$

$$I = V/Z_{ab} = (136 \angle 0^\circ) / (34 \angle -28.07^\circ) = 4 \angle -28.07^\circ \text{ A}$$

- ❖ Example - Application of Source Transformation
Find the Steady state expression for $v(t)$ for the following circuit where:
 $v_1(t) = 240 \cos(4000t + 53.13^\circ) \text{ V}$ and $v_2(t) = 96 \sin(4000t) \text{ V}$



Step1) Convert the signal and V, I and elements to Phasor Domain

$$V_1 = 240 \angle 53.13^\circ = 240 \cos(53.13^\circ) + j240 \sin(53.13^\circ) = 144 + j192$$

$$V_2 = 96 \angle -90^\circ = 96 \cos(-90^\circ) + j96 \sin(-90^\circ) = -j96$$

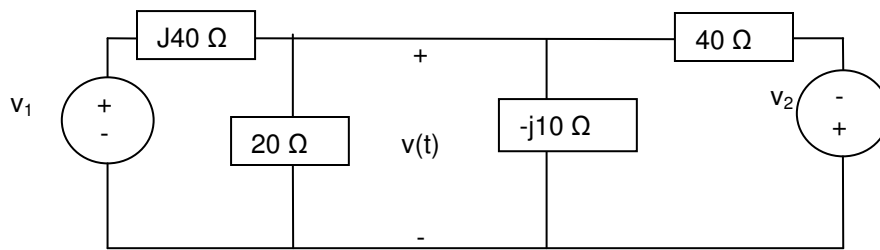
"remember needs to have the signal first in Cosine form $v_2(t) = 96 \cos(4000t - 90^\circ)$ "

$$Z_C = 1/j\omega C = 1/(j*4000*25*10^{-6}) = -j10 \Omega$$

$$Z_L = j\omega L = j*4000*0.01 = j40 \Omega$$

$$Z_R = 40 \Omega \text{ \& } 20 \Omega$$

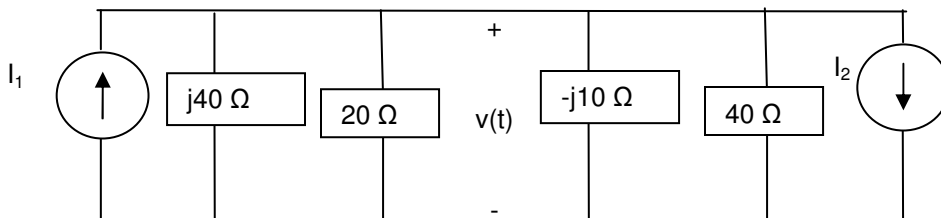
Step2) Redraw circuit in Phasor Domain



Step2) Apply Source Transformation:

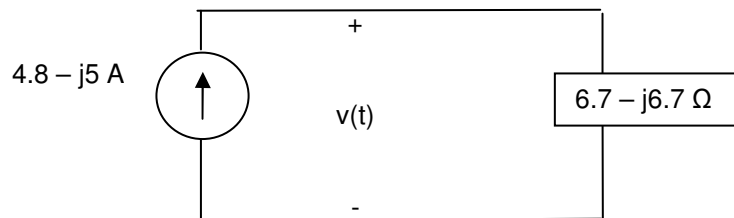
$$V_1 = 144 + j192 \text{ and } Z_1 = j40 \Omega \rightarrow I_1 = V_1/Z_1 = 4.8 - j3.6 \text{ A and } Z_1 = j40 \Omega$$

$$V_2 = -j96 \text{ and } Z_2 = 40 \Omega \rightarrow I_2 = V_2/Z_2 = j2.4 \text{ A and } Z_2 = 40 \Omega$$



Step 3) Using Parallel Impedances Simplification

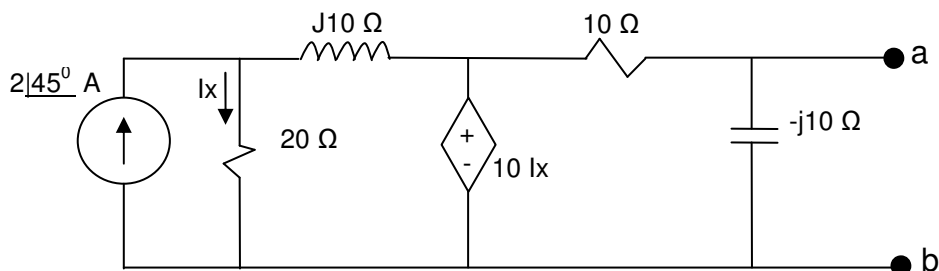
$$Z_{eq} = 1 / (1/j40 + 1/20 - 1/j10 + 1/40) = 6.7 - j6.7$$



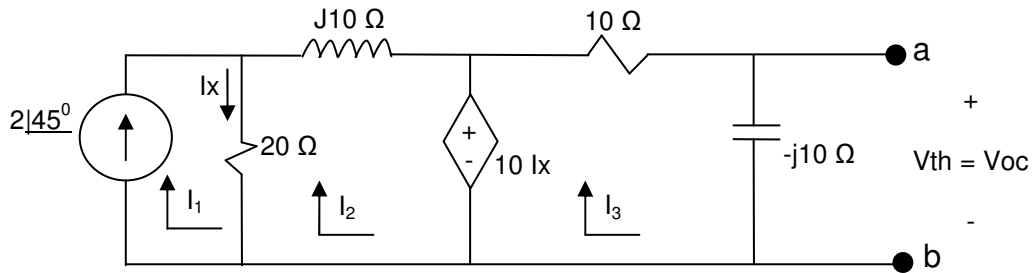
$$V = I * Z_{eq} = (4.8 - j5)(6.7 - j6.7) = -1.3 - j65.7 \text{ V}$$

❖ Example - Application of Thevenin Equivalence

Find Thevenin equivalence with respect to terminals a,b.



Step 1) Find Vth (Open a,b) using Mesh-Current technique



$$\begin{aligned}
 \text{Mesh\#1} &\rightarrow I_1 = 2\angle 45^\circ = 1.4 + j1.4 \\
 \text{Mesh\#2} &\rightarrow 20(I_2 - I_1) + j10I_2 + 10I_x = 0 \rightarrow -20I_1 + 20I_2 + j10I_2 + 10I_1 - 10I_2 = 0 \\
 \text{Mesh\#3} &\rightarrow -10I_x + 10I_3 - j10I_3 = 0 \rightarrow -10I_1 + 10I_2 + 10I_3 - j10I_3 = 0 \\
 \text{Dependent source} &\rightarrow I_x = I_1 - I_2
 \end{aligned}$$

The above four equations can be reduced to two:

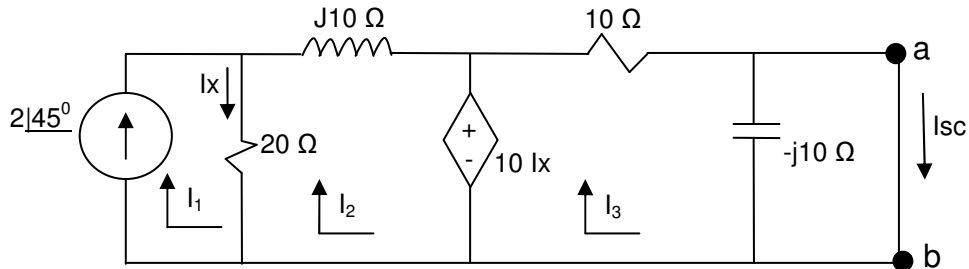
$$\begin{aligned}
 -10I_1 + 10I_2 + j10I_2 &= 0 \rightarrow 30(1.4 + j1.4) + (-30 + j10)I_2 = 0 \rightarrow I_2 = 0.8 + j1.7 \\
 -10I_1 + 10I_2 + 10I_3 - j10I_3 &= 0 \rightarrow -10(1.4 + j1.4) + 10I_2 + (10 - j10)I_3 = 0
 \end{aligned}$$

rewrite last equation with value of I_2

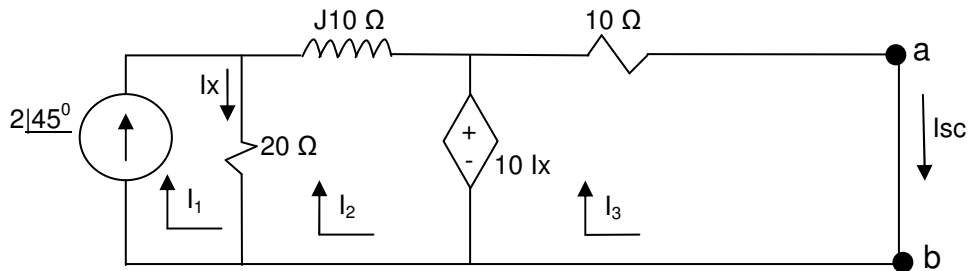
$$-10(1.4 + j1.4) + 10(0.8 + j1.7) + (10 - j10)I_3 = 0 \rightarrow I_3 = (6 - j3) / (10 - j10) = 0.45 + j0.15$$

$$V_{th} = V_{oc} = (-j10)(I_3) = 1.5 - j4.5 \text{ V}$$

Step 2) Find Isc (Short a,b) using Mesh-Current technique



note that the capacitor is in parallel with a short circuit so will not have any effect and can be removed to redraw the circuit as:



$$\begin{aligned}
 \text{Mesh\#1} &\rightarrow I_1 = 2\angle 45^\circ = 1.4 + j1.4 \\
 \text{Mesh\#2} &\rightarrow 20(I_1 - I_2) + j10I_2 + 10I_x = 0 \rightarrow 20I_1 + (-20 + j10)I_2 + 10I_3 = 0 \\
 \text{Mesh\#3} &\rightarrow -10I_x + 10I_3 = 0 \rightarrow I_x = I_3 \\
 \text{Dependent source} &\rightarrow I_x = I_1 - I_2 \rightarrow I_3 = I_1 - I_2 = 1.4 + j1.4 - I_2
 \end{aligned}$$

We can reduce above 4 equations to the following two equations:

$$20(1.4 + j1.4) + (-20 + j10)I_2 + 10I_3 = 0$$

$$I_3 = 1.4 + j1.4 - I_2$$

Plug I_3 into first equation:

$$20(1.4 + j1.4) + (-20 + j10)I_2 + 10(1.4 + j1.4 - I_2) = 0 \rightarrow I_2 = (42 + j42)/(30 - j10)$$

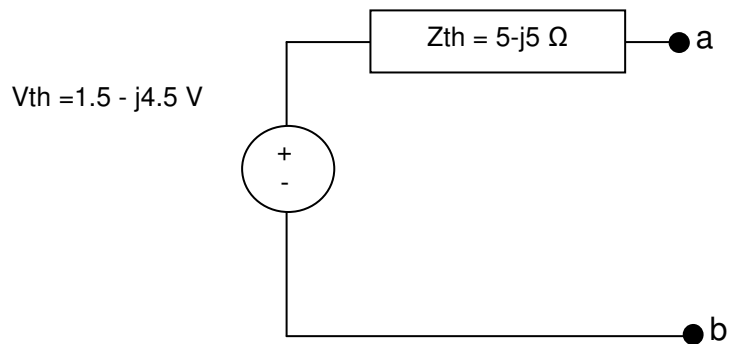
$$\rightarrow I_2 = 0.84 + j1.68$$

$$I_3 = 0.6 - j0.3$$

$$I_{sc} = I_3 = 0.6 - j0.3$$

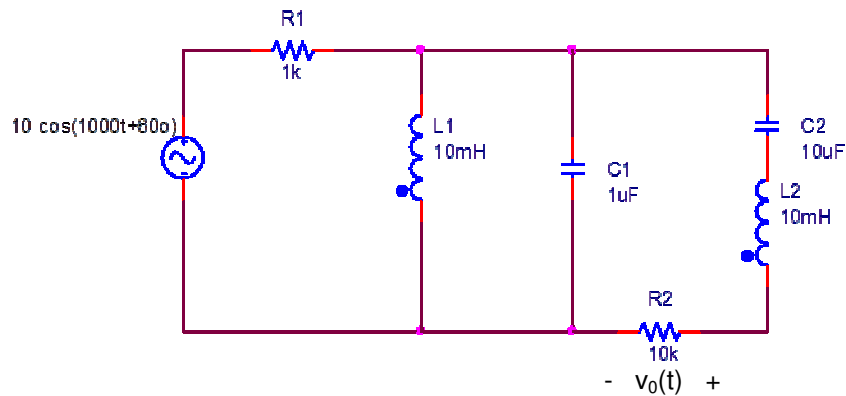
Step 3: Find $Z_{th} = V_{th} / I_{sc}$ & Draw the Thevenin Equivalent

$$Z_{th} = V_{th} / I_{sc} = (1.5 - j4.5) / (0.6 - j0.3) = (1.5 - j4.5) / (0.6 - j0.3) = 5 - j5 \Omega$$



❖ Example – Phasor Domain Analysis

Find value of $V_o(t)$ in the following circuit:



- Solution (Ans: $0.1 \cos(1000t - 0.52)$):

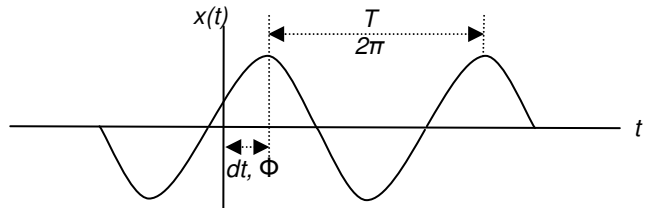
9.7. Summary

❖ Sinusoidal Signal in Time Domain:

$$x(t) = X_M \cos(\omega t + \phi) \quad \text{Where}$$

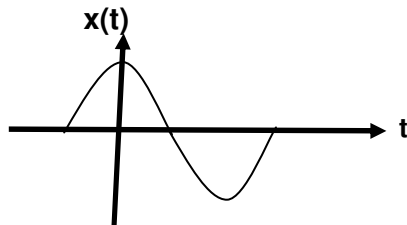
Angular Frequency, $\omega = 2\pi f = 2\pi/T$
 Phase shift, Φ

Linear/Angular relation: $\frac{dt}{T} = \frac{\phi}{2\pi}$



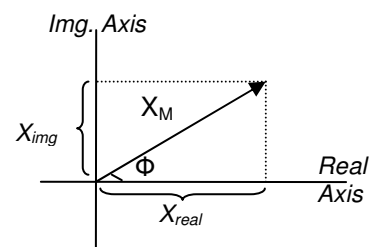
❖ Time-domain to complex format

- In time domain, time is the variable and function is real number.



- $X_M \cos(\omega t + \phi)$

- Complex Coordinate System and time is not a variable.



Rectangular Form:

$$X = X_{real} + jX_{img} = X_M \cos \phi + jX_M \sin \phi$$

Polar Form:

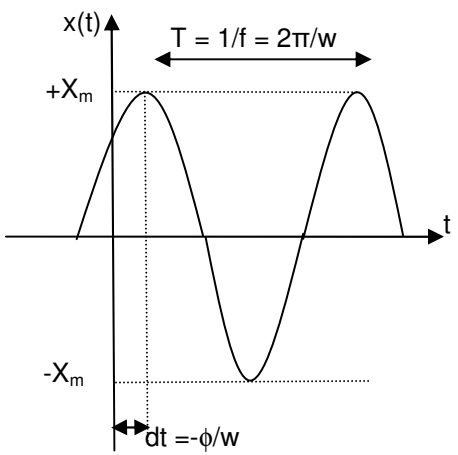
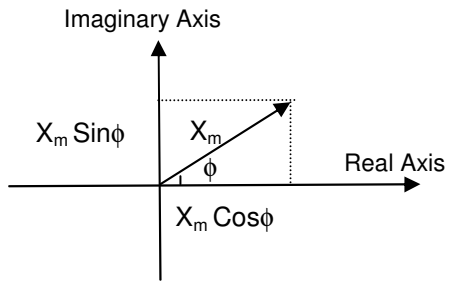
$$X = X_M e^{j\phi} = \sqrt{X_{real}^2 + X_{img}^2} e^{j \tan^{-1}(X_{img}/X_{real})}$$

Angular Form: $X = X_M \angle \phi$

Euler's Identity: $e^{j\theta} = \cos \theta + j \sin \theta$

❖ Time - Phasor domain Transformations

The Phasor Domain Transformation is only valid for Sinusoidal Steady State Operation with single frequency (ω constant).

Time Domain	Phasor Domain
$x(t) = X_m \cos(\omega t + \phi)$ 	<ul style="list-style-type: none"> Polar Form: $X = X_m e^{j\phi}$ Rectangular Form: $X = X_m \cos\phi + jX_m \sin\phi$ Angular Form: $X = X_m \angle \phi^\circ$ 
v(t) & i(t) Relationships $v(t) = R i(t)$ $v(t) = L di(t)/dt$ $i(t) = C dv(t)/dt$	V & I Relationships (Impedance $Z = V/I$ Ohms) $Z = R$ $Z = j\omega L$ $Z = 1/(j\omega C) = -j/\omega C$
Analysis/Modeling Tools Passive convention Resistor simplification KCL KVL Thevenin Equivalent Norton Equivalent $R_L = R_{th} \rightarrow$ Maximum power in Resistive circuit	Analysis/Modeling Tools Passive convention Impedance (Z) simplification KCL KVL Thevenin equivalent Norton equivalent $Z_L = Z_{th} \rightarrow$ Maximum power in RLC circuit
Phasor Domain Problem Solving Steps. <ol style="list-style-type: none"> 1) Transform the circuit to Phasor Domain 2) Apply appropriate analysis/modeling tools to solve the problem 3) Transform the solution back to Time Domain if required 	
Useful Relationships <ul style="list-style-type: none"> Euler's Identity $\rightarrow e^{j\theta} = \cos\theta + j\sin\theta$ or $\cos\theta = \frac{e^{+j\theta} + e^{-j\theta}}{2}$ or $\sin\theta = \frac{e^{+j\theta} - e^{-j\theta}}{2}$ $Z = (R + jX)$ is impedance, R is Resistance, X is Reactance and all are in Ohms $Y = 1/Z$ is referred to as Admittance and Imaginary part of $1/Z$ is referred to as Susceptance $j = \sqrt{-1} = e^{j\pi/2} = -\frac{1}{j}$ $\sin\theta = \cos(\theta - \pi/2)$ 	

9.8. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 9.

9.9. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 10. AC Power Analysis

Key Concepts and Overview

- ❖ AC Power Components
- ❖ Root Mean Squared (RMS) or Effective Power
- ❖ Complex Power
- ❖ Maximum (Average or Real) Power Transfer
- ❖ Additional Resources

10.1 AC Power Components

Similar to DC power calculation, Alternative Current (AC) power can be calculated using the following equation:

$$p(t) = v(t)i(t) \text{ where } v(t) \text{ and } i(t) \text{ are the Instantaneous current}$$

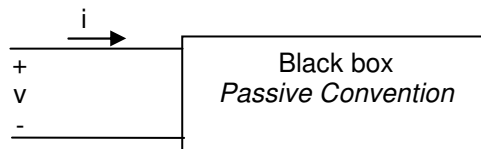
$p(t)$ is the power at time t which is referred to as instantaneous power. Instantaneous power consists of Average (or Real) Power and Magnetizing Reactive Power .

In DC analysis, upper case letter were used to refer to constant value of voltage, current and power (V , I , P) in the circuit. In order to signify dependency of AC voltage, current and power on time (t), these signal parameters are represented using lower case letters (i , v , p). It is common to write p , v and i , instead of writing $p(t)$, $v(t)$ and $i(t)$.

❖ Instantaneous Power, $p(t)$

As mentioned earlier instantaneous power is represented by the following equation and the sign is define by passive convention (current enters the positive voltage terminal indicates positive power):

$$p = v(t) i(t) = v i$$



As discussed in earlier chapters, steady-state sinusoidal circuits represent majority of circuits in use. Therefore, we will continue to focus on these types of circuits. Power Engineering work on energy generation and delivery which has led to a rich set of theories, technology and processes to design and analyze power system in steady-state sinusoidal condition.

The first step is to define the current and voltage equations. As shown in earlier chapters, the general sinusoidal equations for current and voltage are as follows:

$$\begin{aligned} v &= V_m \cos(\omega t + \theta_v) \\ i &= I_m \cos(\omega t + \theta_i) \end{aligned}$$

In power analysis, we are interested in difference between current and voltage phases so instead of having to deal with two general forms. It can be assumed that θ_i is the reference. This assumption lead to simplification of current equation as shown below:

$$\begin{aligned} \text{Current has a phase shift of } (\theta_i - \theta_i) &= 0 \rightarrow i = I_m \cos(\omega t) \\ \text{voltage has a phase shift of } (\theta_v - \theta_i) &\rightarrow v = V_m \cos(\omega t + \theta_v - \theta_i) \end{aligned}$$

Now that current and voltage have been defined, instantaneous power is simply the product of the two parameters:

$$p = v i = I_m V_m \cos(\omega t + \theta_v - \theta_i) \cos(\omega t)$$

We could stop here since given the current and voltage, the power can be calculated using the above equation. But we need to manipulate this equation further to identify two key components for instantaneous power that provides valuable insight.

Applying trigonometric identity $\{\cos(a) \cos(b) = \frac{1}{2} \cos(a - b) + \frac{1}{2} \cos(a + b)\}$, p can be rewritten as:

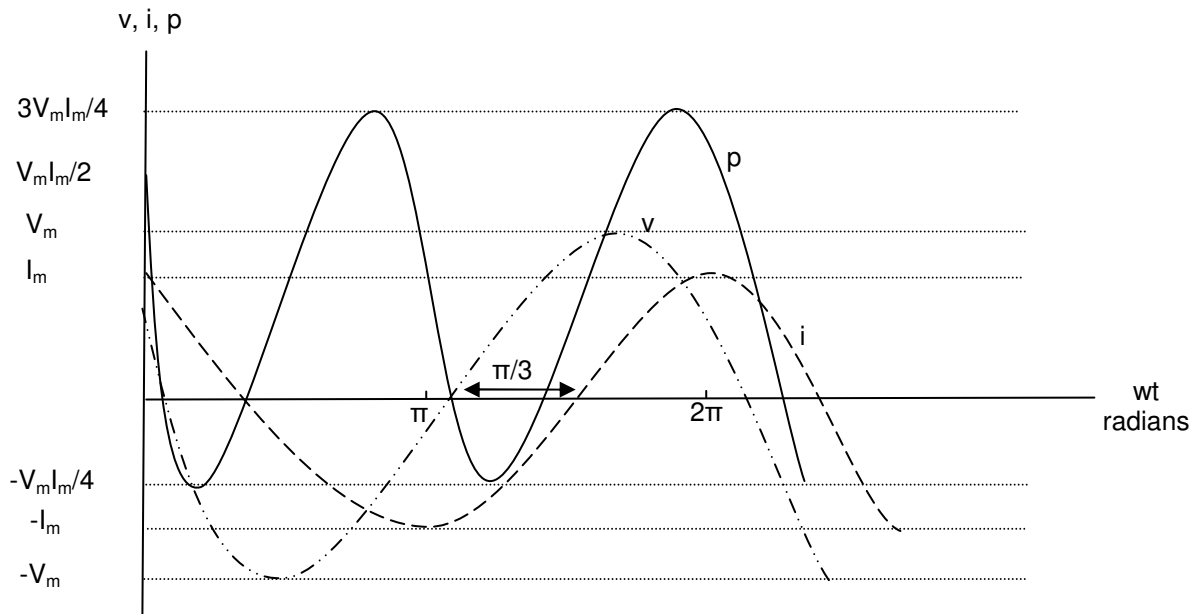
$$p = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(2\omega t + \theta_v - \theta_i)$$

Applying trigonometric identity $\{\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)\}$ to second term results in:

$$p = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \cos(2\omega t) - \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \sin(2\omega t)$$

- Observations:
 - (1) Instantaneous power consists of constant first term and two time-dependent terms with opposing sign.
 - (2) Frequency of instantaneous power is twice the frequency voltage or current.
 - (3) Instantaneous power may be negative for a portion of each cycle (even for completely passive devices. For a completely passive device, the negative power (generating power) means that the energy stored in C or L is being extracted.

At this point, it would be helpful to look at a graphical representation of instantaneous power when current phase shift $\theta_i = 0^\circ$ and voltage phase shift $\theta_v = 60^\circ = \pi/3$.



❖ Average and Reactive Power

Instantaneous power has two distinct components that will be discussed in this section. The Average (or Real) Power and Reactive Power components.

Let's start with the instantaneous equation derived in the previous section:

$$p = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \cos(2\omega t) - \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \sin(2\omega t)$$

The first term is not time-dependent and is referred to as Average or Real Power:

$$P = \text{Average Power or Real Power} = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

The name average power is derived from the fact that it is equal to the average of instantaneous

power over a full period is equal to P. You may want to carry out the following integral to prove that it is true:

$$P = \frac{1}{T} \int_{t_0}^{T+t_0} p dt = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

Additionally, Average Power is the portion of instantaneous electric power that can be converted to non-electric form and visa versa. This is the reason that Average Power is also referred to as Real Power.

By applying the definition of P to the instantaneous power equation, p, can be written as:

$$p = P + P \cos(2\omega t) - \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \sin(2\omega t)$$

The coefficient of $\sin(2\omega t)$ in the third term is referred to as the reactive power, Q:

$$Q = \text{reactive power} = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i)$$

The reactive power is the electric power exchanged between magnetic fields of power supply, capacitors and inductors. Reactive power is never converted to non-electric power.

Utilizing the instantaneous, average and real power, the instantaneous power equation can be rewritten as:

$$p = P + P \cos(2\omega t) - Q \sin(2\omega t) \text{ where}$$

$$P = \text{Average Power or Real Power} = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$$

$$Q = \text{Reactive Power} = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i)$$

As a consumer of electricity, Average Power, P, is the power that does the work of running your motors and heating while Q is power that is moved around in the system between capacitors and inductors. Both consumers and generators of electricity prefer P to be as large as possible and Q to be as small as possible. Later in this chapter, we will discuss how to maximize Average Power.

Power Factor is used to define the relationship between the Average and Reactive power. Power Factor as a measure is important to power industry since the ratio of Average and Reactive power defines the efficiency of the power generation and consumption systems.

The key Power Factor definitions are:

$$\begin{aligned} \text{Power Factor Angle} &= (\theta_v - \theta_i) \\ \text{Power Factor} &= \text{pf} = \cos(\theta_v - \theta_i) \\ \text{Reactive Factor} &= \text{rf} = \sin(\theta_v - \theta_i) \end{aligned}$$

Observations:

- If $(\theta_v - \theta_i) = 0$ then $\text{pf} = \cos(\theta_v - \theta_i) = 1$ and $\text{rf} = \sin(\theta_v - \theta_i) = 0$ which means all the power is Real and non is Reactive. Most efficient system from generator and consumers' point-of-view. Both consumers and generator work toward this ideal situation but is very hard to achieve under variable load.

- Power Factor (pf) is not sufficient to describe the power factor angle since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$ so we have to use the phrase of leading or lagging to qualify it.
 - Lagging power factor implies that current lags the voltage \rightarrow
 $\theta_i = \theta_v - \alpha^\circ \rightarrow (\theta_v - \theta_i) > 0$ "Inductive load"
 - Leading power factor implies that current leads the voltage \rightarrow
 $\theta_i = \theta_v + \alpha^\circ \rightarrow (\theta_v - \theta_i) < 0$ "Capacitive Load"

At this point, let's apply the concepts of instantaneous, Average and Reactive power to the three pure circuits (pure resistive, capacitive and inductive):

- Power in purely resistive circuits
 Since resistive circuit impedance is a real number then Voltage and Current are in phase.
 Therefore $(\theta_v - \theta_i) = 0$ or $\theta_v = \theta_i$ which means:

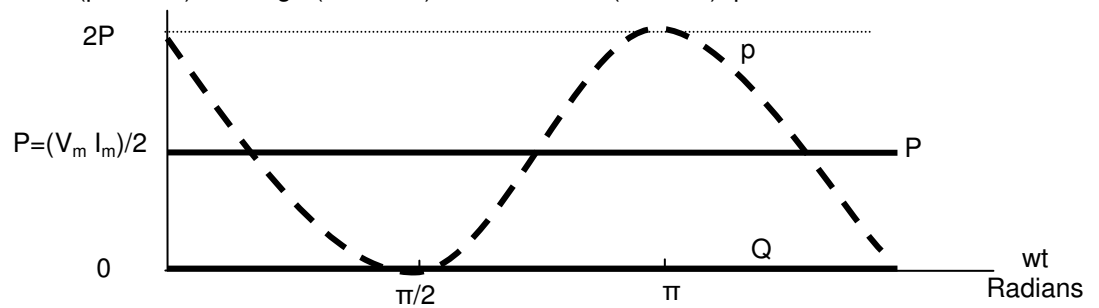
$$p = P + P \cos(2\omega t) \text{ where}$$

$$P = (V_m I_m) / 2$$

$$Q = 0;$$

Graphic representation of Instantaneous, Real, Reactive power

Instantaneous (p, Watts), Average (P, Watts) and Reactive (Q, VAR) power



Observations:

- Instantaneous power is all real
- No negative power (one cannot extract power from Purely Resistive circuit)
- Power's frequency (2wt) is twice the voltage's and current's frequency (wt).

- Power for Purely Inductive Circuits
 Impedence of a purely inductive circuit is $\{j\omega L = \omega L e^{j\pi/2}\}$ which means the current lags the voltage by $\pi/2$ therefore $(\theta_v - \theta_i) = \pi/2 = 90^\circ$

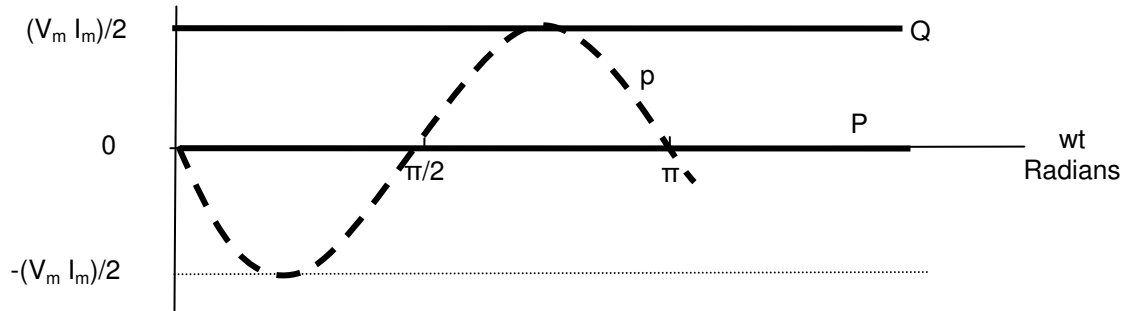
$$p = -Q \sin(2\omega t) \text{ where}$$

$$Q = (V_m I_m) / 2$$

$$P = 0;$$

Graphic representation of Instantaneous, Real, Reactive power

Instantaneous (p, Watts), Average (P, W) and Reactive (Q, VAR) power



Observations:

- Instantaneous power has no real component (real power is zero) meaning no transforming from electrical to non-electric energy takes place.
 - The name reactive comes from the fact that this is the only power from reactive elements such as Capacitors and inductors.
- Although average power and reactive power carry the same dimensions, we use different unit to distinguish between the two types of power:
 - Watt (W) is used as the unit for real or average powers and instantaneous power
 - VAR (Volt-Amp Reactive) is used as the unit for reactive powers
- Circuit continuously exchange power at power's frequency (twice voltage or current frequency, $2\omega t$)
 - p is positive, energy is being stored in the magnetic field associated with the inductor
 - p is negative, energy is being extracted from the magnetic field

➤ Power for Purely Capacitive Circuits

Impedence of a purely Capacitive Circuit is $\{1/j\omega C = e^{-j\pi/2}/\omega C\}$ which means the current leads the Voltage by $\pi/2$ therefore $(\theta_v - \theta_i) = -\pi/2 = -90^\circ$

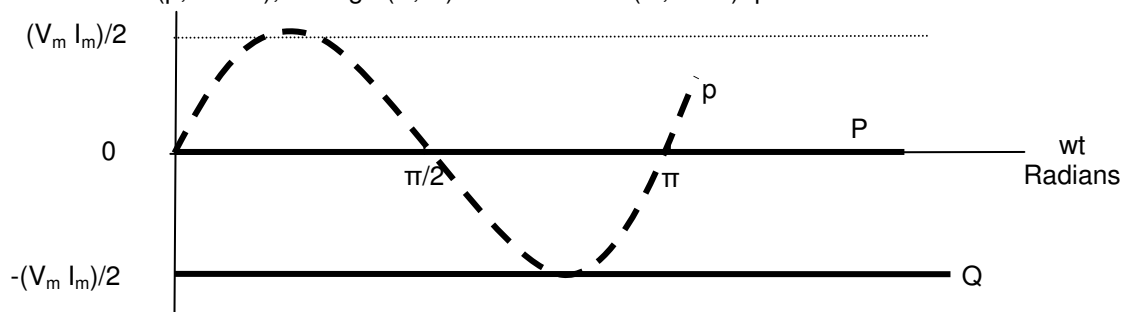
$$p = Q \sin(2\omega t)$$

$$Q = -(V_m I_m) / 2$$

$$P = 0;$$

Graphic representation of Instantaneous, Real, Reactive power

Instantaneous (p, Watts), Average (P, W) and Reactive (Q, VAR) power



Observations

- Instantaneous power has no real component (real power is zero) meaning no transforming from electrical to non-electric energy takes place.
- Circuit continuously exchange power at power's frequency (twice voltage or current frequency, $2\omega t$)
 - p is positive, energy is being stored in the electric field associated with the capacitors

(b) p is negative, energy is being extracted from the electric field

Power industry recognizes the algebraic sign of + for inductors and – for capacitors by saying:

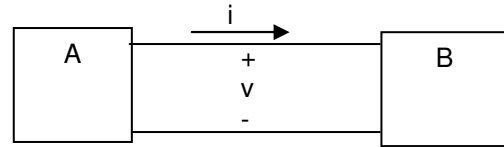
- Inductors demand (or absorb) magnetizing VARs.
- Capacitors furnish (or deliver) magnetizing VARs.

❖ Example –AC Power Analysis

Calculate the Real, Reactive Power transferring between network A and B, state whether the power flow is from A to B or vice-versa. Also state whether magnetizing VARs are being transferred from A to B or vice versa. Use the following voltage and current equations, and figure.

$$v = 100 \cos(\omega t - 35^\circ) \text{ V}$$

$$i = 20 \cos(\omega t + 155^\circ) \text{ A}$$



Solution:

$$\text{Power Factor Angle} = (\theta_v - \theta_i) = (-35 - (155)) = -190^\circ$$

$$\text{Real Power} = P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) = \frac{(20)(100)}{2} \cos(-190) = -984.8 \text{ W (B to A)}$$

$$\text{Reactive Power} = Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) = 173.6 \text{ VAR (A to B)}$$

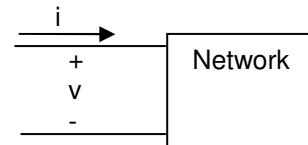
Note: B is configured as passive device (Current flows to positive terminal) in the diagram. Therefore, if the power is positive then B is consuming and A is generating power

❖ Example -- AC Power Analysis

Compute the power factor and the reactive factor for the network inside the box, with the following current and voltage.

$$v = 200 \cos(\omega t + 15^\circ) \text{ V}$$

$$i = 95 \sin(\omega t - 15^\circ) \text{ A}$$



SOLUTION:

First step is to convert i to standard cosine form $\rightarrow i = 95 \cos(\omega t - 15^\circ - 90^\circ) = 95 \cos(\omega t - 105^\circ) \text{ A}$
Therefore:

$$\theta_v = 15$$

$$\theta_i = -105$$

$$\text{Power Factor Angle} = (\theta_v - \theta_i) = 15 - (-105) = 120^\circ$$

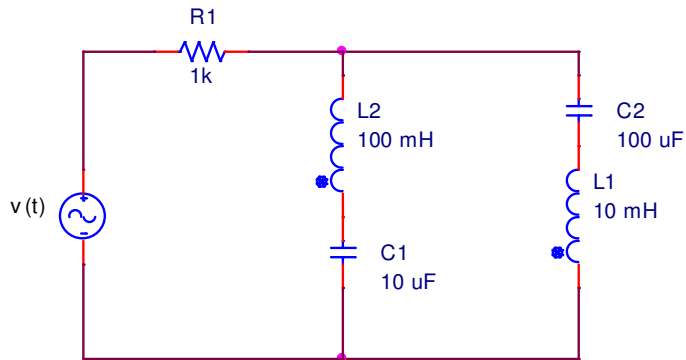
$$\text{Power Factor} = \text{pf} = \cos(\theta_v - \theta_i) = \cos(120) = -0.5$$

$$\text{Reactive Factor} = \text{rf} = \sin(\theta_v - \theta_i) = \sin(120) = 0.866$$

Lagging power factor since $(\theta_v - \theta_i) = +120^\circ > 0$ which means it is an inductive load.

❖ Example – AC Power Analysis

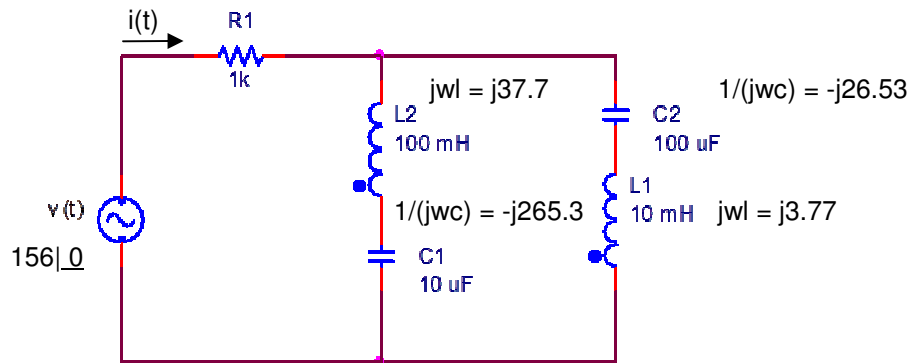
$v(t)$ is a sinusoidal signal with 60 Hz frequency and maximum voltage of 156 V in the following circuit.



Find Power Factor (pf), Real Power (P), Reactive Power (Q).

SOLUTION:

Covert circuit components to Phasor Domain ($\omega = 2\pi f = 120\pi$):



$$Z_{eq} = 1000 + (-j265.3 + j37.7) \parallel (-j26.53 + j3.77) = 1000 - j20.7 = 1000.2 e^{-j1.2}$$

$$\text{Power Factor} = \text{pf} = \cos(\text{power factor Angle}) = \cos(\Theta_v - \Theta_i) = \cos(-1.2) = 1 \text{ Leading}$$

$$I_m = V_m/Z_{eq} = 156/1000 e^{-j1.2} = 0.156 \angle -1.2^\circ$$

$$\text{Real Power} = P = (V_m I_m / 2) \cos(\Theta_v - \Theta_i) = 12.2 \text{ W}$$

$$\text{Reactive Power} = Q = (V_m I_m / 2) \sin(\Theta_v - \Theta_i) = -0.25 \text{ VARs}$$

10.2. Root Mean Squared (RMS) or Effective Power

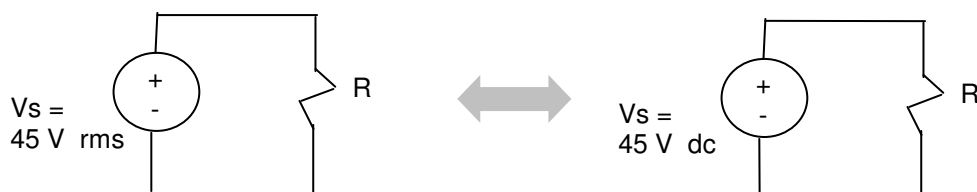
Root Mean Squared (RMS) is used to calculate the sustained power consumption or generation ability of a device. Instantaneous power changes with time from a maximum positive value to a minimum position value. In a steady-state sinusoidal system, RMS power is the accepted method of calculating the sustained power and it is commonly referred to as the effective value also.

In order to calculate the RMS or Effective power, we have to start by calculating the RMS or Effective voltage and current for a sinusoidal steady-state circuit:

$$V_{rms} = V_{eff} = \sqrt{\frac{1}{T} \int_{t_0}^{T+t_0} v^2 dt} = \sqrt{\frac{1}{T} \int_T V_m^2 \cos^2(\omega t + \phi_v) dt} = \frac{V_m}{\sqrt{2}}$$

$$I_{rms} = I_{eff} = \sqrt{\frac{1}{T} \int_{t_0}^{T+t_0} i^2 dt} = \sqrt{\frac{1}{T} \int_T I_m^2 \cos^2(\omega t + \phi_i) dt} = \frac{I_m}{\sqrt{2}}$$

V_{rms} or V_{eff} deliver the same power to a resistive load as a DC voltage of the same magnitude as shown in the following figure:



The effective value of v_s (45 V rms) deliver the same power to R as the dc voltage v_s (45 V dc)

The effective values of sinusoidal signal in power calculations is so widely used that voltage ratings of circuits and equipments in power utilization is always given in terms of rms value. For example the voltage of electricity in typical US homes is stated as $v=110$ V which means that the RMS or Effective voltage is 110 V. The Peak Voltage value may be calculated using the following equation:

$$\text{Peak Voltage} = V_{max} = \sqrt{2} V_{rms} = 156 \text{ V}$$

If we need to find the effective current through a 100 Watts light bulb in a typically US home Then:

$$I_{eff} = P / V_{eff} = 100/110 = 0.91 \text{ A rms}$$

Of course the peak current value may calculated using the following equation:

$$\text{Peak Current} = I_{max} = \sqrt{2} I_{rms} = 1.29 \text{ A}$$

The definition of RMS value can be applied to the Average and Reactive Power as shown below:

$$P = \text{average power} = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) = V_{eff} I_{eff} \cos(\theta_v - \theta_i)$$

$$Q = \text{reactive power} = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \sin(\theta_v - \theta_i) = V_{eff} I_{eff} \sin(\theta_v - \theta_i)$$

❖ Example - Power consumption and rating

A typical system is capable of generating or consuming peak power at many times its RMS power rating. Furthermore, Average Power is the portion of power that can be converted to non-electrical power and used. It is expected and recommended that electrical devices be rated in terms of Effective or RMS Average power, P.

Find the number of computers that can run off a circuit breaker capable of $I_{eff}=15$ Amp at $V_{eff}=110$ v. Each computer consists of a SPU (RMS Average Power of 200 watts) and a LCD display (Average Power of 100 watts).

$$P_{\text{one computer}} = 200 + 100 = 300 \text{ Watt}$$

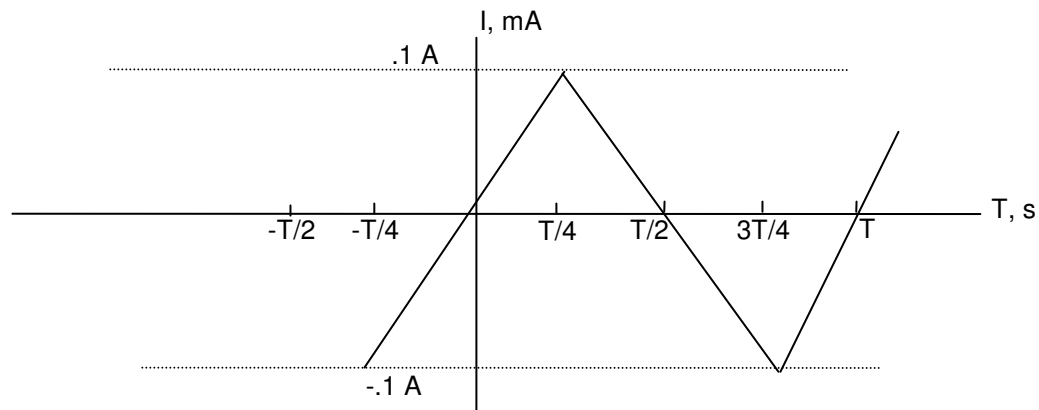
$$P_{\text{available}} = I_{eff} * V = 15 * 110 = 1650 \text{ Watt}$$

$$\# \text{ of computer} = \text{integer} \{ P_{\text{available}} / P_{\text{one computer and monitor}} \} = \text{integer} \{ 1650/300 \} = 5$$

Note: the instance power may be an issue if all system are turned on at the same time since turn-on power requirements are higher than steady state usage.

❖ Example – RMS Value for non-Sinusoidal Signals

A source is capable of delivering a periodic triangular signal with peak current of 100 mA. Find the average power delivered to a 4 K Ω resistor.



$$I_{rms} = I_{eff} = \sqrt{\frac{1}{T} \int_0^{T+0} i^2 dt} = \sqrt{\frac{4}{T} \int_0^{T/4} i^2 dt}$$

Note: only need to calculate 0 to T/4 then multiply by 4 (simplification)

The line equation representing $i(t)$ $0 \leq t \leq T/4 \rightarrow i = (0.1/(T/4)) t = (0.4/T) t$

$$I_{rms} = \sqrt{\frac{4}{T} \int_0^{T/4} \{(0.4/T)t\}^2 dt} = \sqrt{\frac{4(0.4)^2}{T^3} \left(\frac{t^3}{3} \Big|_0^{T/4}\right)} = \sqrt{\frac{4(0.4)^2}{T^3} \left(\frac{T^3}{(3)(4^3)}\right)} = \sqrt{0.003} \quad A$$

Note: Period is cancels out and does not effect the value of I_{rms} .

$$P = I_{eff}^2 R = (0.003) * 4000 = 12 \text{ W}$$

10.3. Complex Power

As we discussed in earlier Chapter, steady-state sinusoidal analysis and associated mathematics may be simplified by using complex number and Phasor representation. The same is true for AC power analysis. AC power may be represented using complex number form which is referred to as Complex Power and is represented by letter S and defined below:

$$\text{Complex Power} = S = P + jQ \quad \text{in Volt-Amp (VA)}$$

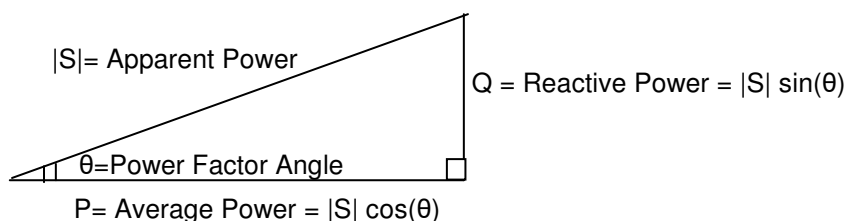
where:

$$P = \text{Average or Real Power} = \text{Real part of } \{S\} \text{ in Watts (W)}$$

$$Q = \text{Reactive Power} = \text{Imaginary part of } \{S\} \text{ in Volt-Amp Reactive (VAR)}$$

Using the Complex Power offers two advantages:

- (1) Use of complex representation of V, I and Z to calculate power
- (2) Ability to relate power components using trigonometry as shown below (Power Rectangle):



Observations:

- (1) Angle θ in the above diagram is the Power Factor Angle ($\theta_v - \theta_i$). Below is the proof:

$$\tan \theta = \frac{Q}{P} = \frac{(V_m I_m / 2) \sin(\theta_v - \theta_i)}{(V_m I_m / 2) \cos(\theta_v - \theta_i)} = \tan(\theta_v - \theta_i) \rightarrow \theta = (\theta_v - \theta_i)$$

- (2) The magnitude of the complex power is called Apparent Power:

$$|S| = \sqrt{P^2 + Q^2}$$

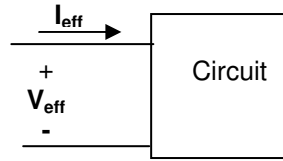
Apparent Power like Complex Power is measured in Volt-Amp (VA). $|S|$ is more important than real power when discussing devices that have a leading or lagging Power Factor (Capacitance or inductive load). In other words are not pure resistive circuits.

In general Average Power represents the usable output of the transferred energy. While the apparent power represents the volt-amp capacity required to supply the desired Average Power. Only in a resistive circuit, the capacity to supply Average Power and Apparent Power are the same. In inductive loads (most common), we need to correct for the lag power factor to maximize average power delivered.

- (3) Power Factor, $PF = \cos(\theta) = P / |S|$

Complex Power, S, calculation utilizes the learning from the Phasor section and definition of Average and Reactive power earlier in this Chapter.

Starting with the following definition of a device where effective current and voltage are represented in complex form:



$$S = P + jQ = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + j \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) = \frac{V_m I_m}{2} [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)]$$

Apply Euler's identity $e^{ja} = \cos(a) + j \sin(a)$

$$S = \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} e^{j(\theta_v - \theta_i)} = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \angle(\theta_v - \theta_i)$$

In Steady – State Sinusoidal circuit $\Rightarrow I_{eff} = I_{rms} = \frac{I_m}{\sqrt{2}} \quad \& \quad V_{eff} = V_{rms} = \frac{V_m}{\sqrt{2}}$

$$S = V_{eff} I_{eff} \angle(\theta_v - \theta_i)$$

OR

$$S = \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} = V_{eff} I_{eff} \angle(\theta_v - \theta_i) = \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)}$$

Also it is useful to write complex power in term of complex I and V directly as shown below:

$$S = V_{eff} I_{eff} \angle(\theta_v - \theta_i) = V_{eff} I_{eff} e^{j(\theta_v - \theta_i)} = V_{eff} e^{j(\theta_v)} I_{eff} e^{-j(\theta_i)}$$

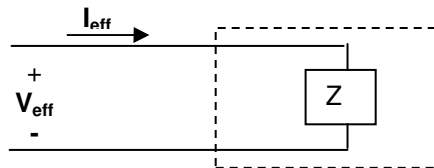
Here is a summary of Complex Power Calculation:

$$\begin{aligned} S = P + jQ &= V_{eff} I_{eff} \angle(\theta_v - \theta_i) = V_{eff} I_{eff} e^{j(\theta_v - \theta_i)} = V_{eff} e^{j(\theta_v)} I_{eff} e^{-j(\theta_i)} = \frac{1}{2} V_m e^{j(\theta_v)} I_m e^{-j(\theta_i)} \\ &= \frac{1}{2} V \cdot I^* = V_{eff} I_{eff}^* = Z I_{eff} \cdot I_{eff}^* = \frac{|V_{eff}|^2}{Z^*} = |I_{eff}|^2 Z \end{aligned}$$

Note:

- 1) Bold **I** & **V** represent max.current and voltage and **I_{eff}** & **V_{eff}** represent the effective or RMS values.
- 2) The “*” indicates conjugate which means imaginary portion is multiplied by negative (or replace j with -j)

Using the Impedance equation ($Z = R + jX$) for a typical device shown below, we can develop an additional set of Complex Power equations:



By applying $\mathbf{V}_{\text{eff}} = \mathbf{Z} \mathbf{I}_{\text{eff}}$ to $\mathbf{S} = \mathbf{V}_{\text{eff}} \mathbf{I}_{\text{eff}}^*$

Complex Power may be written as:

$$\mathbf{S} = \mathbf{Z} \mathbf{I}_{\text{eff}} \mathbf{I}_{\text{eff}}^*$$

We know that product of a complex number with its conjugate $KK^* = (a+jb)(a+jb)^* = (a+jb)(a-jb) = a^2 + b^2 \rightarrow$

$$\sqrt{a^2 + b^2} = |K|$$

Therefore

$$\mathbf{S} = \mathbf{Z} |\mathbf{I}_{\text{eff}}|^2 = |\mathbf{I}_{\text{eff}}|^2 (R + jX) = |\mathbf{I}_{\text{eff}}|^2 R + j|\mathbf{I}_{\text{eff}}|^2 X = P + jQ$$

Which means:

$$P = |\mathbf{I}_{\text{eff}}|^2 R = \frac{1}{2} I_m^2 R$$

$$Q = |\mathbf{I}_{\text{eff}}|^2 X = \frac{1}{2} I_m^2 X$$

If the circuit is purely inductive then Reactance X is positive and if the circuit is pure capacitance then Reactance X will be negative.

Of course the Complex Power may be also written in terms of voltage using similar steps as show here:

$$\text{By applying } \mathbf{I}_{\text{eff}} = \mathbf{V}_{\text{eff}} / \mathbf{Z} \text{ to } \mathbf{S} = \mathbf{V}_{\text{eff}} \mathbf{I}_{\text{eff}}^* \rightarrow \mathbf{S} = \mathbf{V}_{\text{eff}} (\mathbf{V}_{\text{eff}} / \mathbf{Z})^*$$

$$\mathbf{S} = |\mathbf{V}_{\text{eff}}|^2 / \mathbf{Z}^* = P + jQ$$

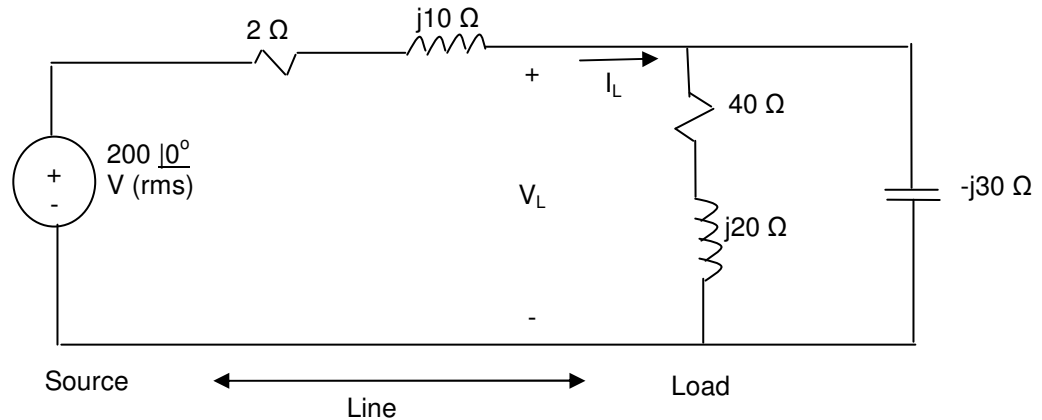
$$\text{If } Z \text{ is pure resistive } \rightarrow P = |\mathbf{V}_{\text{eff}}|^2 / R$$

$$\text{if } Z \text{ is pure reactive } \rightarrow Q = |\mathbf{V}_{\text{eff}}|^2 / X \quad \text{"note } X \text{ is positive for inductor and negative for capacitor"}$$

❖ Example – Complex power Calculation

The following circuit is an electrical power and transmission model. For the following circuit find:

- The RMS Phasor V_L and I_L .
 - The Average Power and magnetizing Reactive Power delivered by the source to the load.
 - Draw the Power Triangle shows Apparent, Average and Reactive Power delivered to the load.
- Note: reactive power is magnetizing so magnetizing Reactive Power is the same as Reactive Power.*



Solution:

- a) RMS Phasor V_L and I_L

Calculate Load equivalent Impedance $\rightarrow Z_{Leq} = 1/(1/(40+j20) + 1/(-j30)) = 21.2 - j24.7$
 Voltage source is already in RMS value $\rightarrow V = 200 \angle 0^\circ$ V (rms)

$$I_L = V / Z_t = (200) / (21.2 - j24.7 + 2 + j10) = 6.2 + j3.9 = 7.2 \angle 32.4^\circ \text{ A (rms)}$$

$$V_L = I_L Z_L = (6.2 + j3.9) (21.2 - j24.7) = 223.8 - j70.5 = 238.42 \angle -17.2^\circ \text{ V (rms)}$$

- b) P & Q delivered to the load

$$S = V_L \dot{I}_L^* = (223.8 - j70.5)(6.2 - j3.9) = 16662.5 + j435 \text{ VA}$$

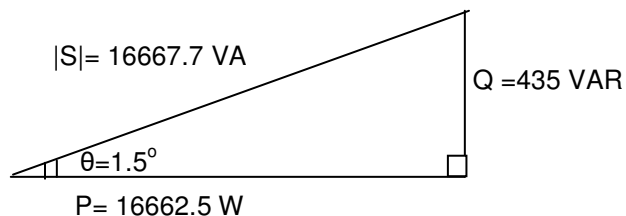
$$\text{Average Power} = P = 16662.5 \text{ W}$$

$$\text{Magnetizing Reactive Power} = Q = 435 \text{ VAR}$$

- c) Power Rectangle

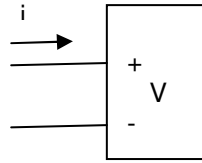
$$\text{Power Factor Angle} = \theta = \tan^{-1} \frac{Q}{P} = \tan^{-1} \frac{435}{16662.5} = 1.5^\circ$$

$$\text{Apparent Power} = |S| = \sqrt{P^2 + Q^2} = 16667.7 \text{ VA}$$



- ❖ Example – Complex Power
For the following device:

$$i(t) = 25 \cos(\omega t + 10^\circ)$$
$$v(t) = 100 \cos(\omega t + 55^\circ)$$



find:

- (a) Real Power
- (b) Reactive Power
- (c) Power Factor
- (d) Instantaneous Power
- (e) Complex power in all three forms (Rect., Polar & Angular)

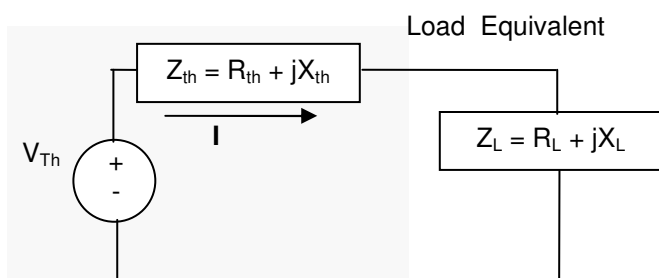
➤ Solution:

10.4. Maximum (Average or Real) Power Transfer

As it was discussed earlier, Average Power is the portion of power that can be converted to non-electrical power such as kinetic or thermal energy. Therefore the goal of energy consumer and generators is to deliver the maximum power to the load.

In this section, the requirement for maximum is explored for an AC circuit. The process is similar to the DC and resistive circuit with the difference that impedance (Z), current (I) and Voltage (v) may be complex numbers instead of real numbers.

A model for power generation and consumption is shown below. Transmission line impedance can be modeled as part of source impedance, load impedance or separately.



Thevenin Equivalent of Generator

Note: V_{th} and I are rms value

Maximum Average Power, P , is transferred when

$$Z_L = Z_{th}^*$$

"*" is Conjugate $\rightarrow (a + jb)^* = (a - jb)$

The key finding here is that for maximum average power to transfer in a Steady-State Sinusoidal condition, load impedance must be equal to the conjugate of Thevenin impedance ($Z_L = Z_{th}^*$).

Now that the answer is known, let's take a look at the proof:

$$\text{KVL} \rightarrow I = \frac{V_{th}}{(R_{th} + R_L) + j(X_{th} + X_L)}$$

$$P = |I|^2 R_L = \frac{|V_{th}|^2 R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$

In order to find the maximum, take a derivative of P in-term of R_L and X_L . Since load is the only variable in this situation.

$$\frac{\partial P}{\partial X_L} = 0 \Rightarrow X_L = -X_{th}$$

$$\frac{\partial P}{\partial R_L} = 0 \Rightarrow R_L = \sqrt{R_{th}^2 + (X_L + X_{th})^2} \Rightarrow R_L = R_{th}$$

Combining the above two conditions we have $\Rightarrow Z_{th} = Z_L^*$

The next question to explore is if the condition ($Z_L = Z_{th}^*$).exists then what is the maximum average power transferred? To answer this question substitute ($R_L = R_{th}$ and $X_L = -X_{th}$) in the earlier Average Power equation:

$$P = |I|^2 R_L = \frac{|V_{th}|^2 R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$

$$P_{\max} = \frac{|V_{th}|^2 R_L}{(R_L + R_L)^2 + j(-X_L + X_L)^2}$$

$$P_{\max} = \frac{|V_{th}|^2}{4R_L} \quad \text{for } V_{th} \text{ rms value}$$

or

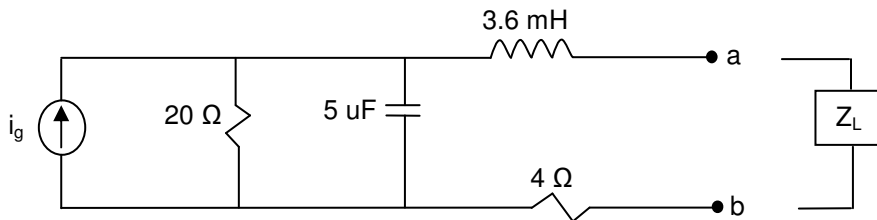
$$P_{\max} = \frac{V_m^2}{8R_L} \quad \text{for max } V_{th} \text{ amplitude since } V_{rms} = V_m / \sqrt{2}$$

There are times that the designer does not have full control over the load so we need to explore approaches to maximizing power transfer under restrictive load conditions. Here are three possible sets of restrictions and associated approaches:

- R_L and X_L may be restricted to a limited range so that you cannot achieve maximum power transfer condition $Z_{th} = Z_L^*$
In this case, the designer is expected to get as close as possible to maximum Average Power by adjusting:
 - (1) Adjust X_L so that it is as close to $(-X_{th})$ as possible
 - (2) Adjust R_L so that it is as close to $\sqrt{R_{th}^2 + (X_L + X_{th})^2}$ as possible.
- Magnitude of Z_L can be varied but its phase angle cannot
In this case, the greatest Average Power is transferred to the load when the magnitude of Z_L is set equal to magnitude of Z_{th} . $\rightarrow |Z_L| = |Z_{th}|$
- Purely Resistive Load
In this case maximum Average Power is transferred when $R_L = R_{th}$

❖ Example - Maximum Average (Real) Power

The source current in the following circuit is $i_g = 3\cos(5000t)$ A.



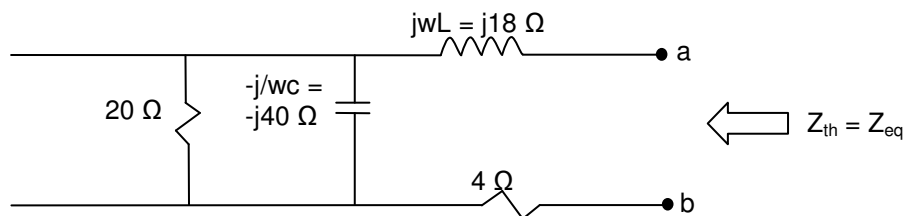
- What load impedance at terminals a and b enable maximum Power Transfer?
- What is the Average Power transferred to the impedance from part a?
- Assume that the load is restricted to pure resistance.
What size load resistor results in the maximum average power transferred?
- What is the Average Power Transferred to the load for part C?

Solution:

Part a) Load impedance for Maximum Power Transfer?

Start by finding the Impedance of the Thevenin Equivalent at $\omega=5000$ rad/s

Redraw the circuit in Phasor domain and deactivate the current source $i_g (=0, \text{open})$



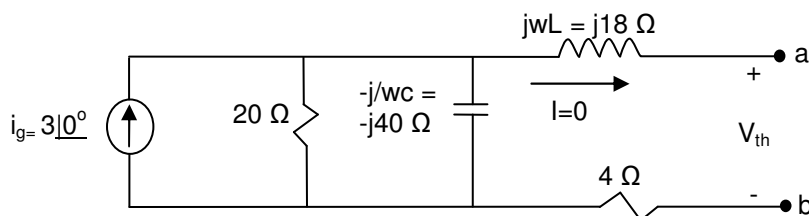
$$Z_{th} = 4 + (20 \parallel (-j40)) + j18 = 20 + j10$$

In order for the maximum power to transfer we have to have: $Z_{th} = Z_L^*$

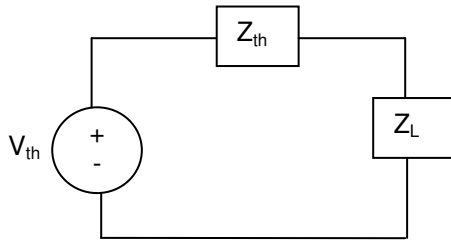
Therefore $\rightarrow Z_L = Z_{th}^* = (20 + j10)^* = 20 - j10$

Part b) What is the average power transferred to the impedance from first part?

Now we have to find $V_{th} = V_{ab-\text{open}}$



$$V_{th} = i_g (20 \parallel -j40) = 3 (16 - 8j) = 53.7 \angle 26.6^\circ \text{ V}$$



$$I_m = \frac{V_{th}}{Z_{th} + Z_L} = \frac{53.67 \angle -26.57^\circ}{40} = 1.34 \angle -26.57^\circ$$

$$I_{eff} = \frac{I_m}{\sqrt{2}} = \frac{1.34}{\sqrt{2}}$$

$$P_{max} = (I_{eff})^2 R_L = \left(\frac{1.34}{\sqrt{2}}\right)^2 20 = 17.96 \text{ W}$$

Another option would be to use $P_{max} = |V_{th}|^2 / (4R_L)$ to solve the problem.

Part c) Assume that the load is restricted to pure resistance, Find R_L ?

The two conditions for maximum power transfer are:

$$\frac{\partial P}{\partial X_L} = 0 \Rightarrow X_L = -X_{th} \leftarrow \text{we have no control over this since } X_L = 0 \text{ given.}$$

$$\frac{\partial P}{\partial R_L} = 0 \Rightarrow R_L = \sqrt{R_{th}^2 + (X_L + X_{th})^2} \leftarrow \text{we have } Z_{th} = 20 + j10$$

$$R_L = \sqrt{20^2 + (0 + 10)^2} = 22.36 \, \Omega \text{ Maximizes the average power transfer to load}$$

Part d) What is the Average Power Transferred with R_L from part c?

$$I_m = \frac{V_{th}}{Z_{th} + Z_L} = \frac{53.67 \angle -26.57^\circ}{42.36 + j0} = 1.23 \angle -39.85^\circ$$

$$I_{eff} = \frac{I_m}{\sqrt{2}} = \frac{1.23}{\sqrt{2}}$$

$$P_{max} = (I_{eff})^2 R_L = \left(\frac{1.23}{\sqrt{2}}\right)^2 22.36 = 16.91 \text{ W}$$

10.5. Summary

❖ Instantaneous Power

$$p(t) = v(t)i(t) = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \cos(2\omega t) - \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \sin(2\omega t)$$

❖ Power Factor

$$\text{Power Factor Angle} = (\theta_v - \theta_i)$$

$$\text{Power Factor} = \text{pf} = \cos(\theta_v - \theta_i)$$

$$\text{Reactive Factor} = \text{rf} = \sin(\theta_v - \theta_i)$$

❖ RMS or Effective Value

$$V_{rms} = V_{eff} = \sqrt{\frac{1}{T} \int_0^T v^2 dt} = \sqrt{\frac{1}{T} \int_0^T V_m^2 \cos^2(\omega t + \phi_v) dt} = \frac{V_m}{\sqrt{2}}$$

$$I_{rms} = I_{eff} = \sqrt{\frac{1}{T} \int_0^T i^2 dt} = \sqrt{\frac{1}{T} \int_0^T I_m^2 \cos^2(\omega t + \phi_i) dt} = \frac{I_m}{\sqrt{2}}$$

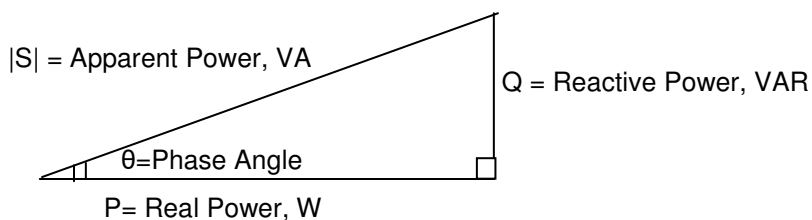
❖ Average and Reactive Power

$$P = \text{average power} = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) = V_{eff} I_{eff} \cos(\theta_v - \theta_i)$$

$$Q = \text{reactive power} = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \sin(\theta_v - \theta_i) = V_{eff} I_{eff} \sin(\theta_v - \theta_i)$$

❖ Complex Power

$$S = P + jQ = V_{eff} I_{eff} \angle (\theta_v - \theta_i) = V_{eff} I_{eff} e^{j(\theta_v - \theta_i)} = V_{eff} e^{j(\theta_v)} I_{eff} e^{-j(\theta_i)} = \frac{1}{2} V_m e^{j(\theta_v)} I_m e^{-j(\theta_i)}$$



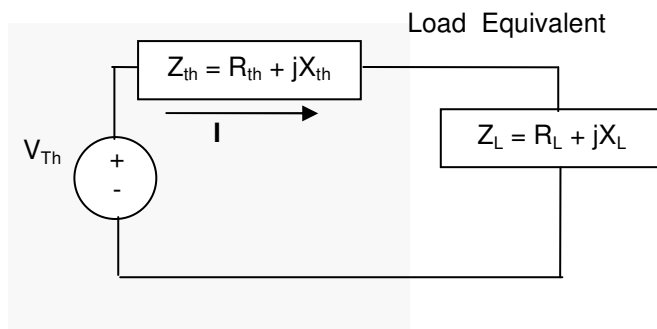
$$\text{Power Factor Angle} = \theta = \tan^{-1} \frac{Q}{P}$$

$$\text{Apparent Power} = |S| = \sqrt{P^2 + Q^2}$$

❖ Complex Power Derivations

$$\mathbf{S} = \mathbf{V}_{eff} \mathbf{I}_{eff}^* = \mathbf{Z} \mathbf{I}_{eff} \mathbf{I}_{eff}^* = |\mathbf{V}_{eff}|^2 / \mathbf{Z}^*$$

❖ Maximum Power



Maximum Average Power, P , is transferred when

$$Z_L = Z_{th}^*$$

"" is Conjugate $\rightarrow (a + jb)^* = (a - jb)$

10.6. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 10.

10.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 11. Three-Phase Circuits

Key Concepts and Overview

- ❖ Introduction
- ❖ Balanced Three-Phase Voltages
- ❖ Three-Phase Voltage Sources
- ❖ Analysis of the Wye-Wye ($\mathbf{Y} \rightarrow \mathbf{Y}$) and Wye-Delta ($\mathbf{Y} \rightarrow \Delta$) Circuits
- ❖ Balanced Three-Phase Circuit Power Analysis
- ❖ Additional Resources

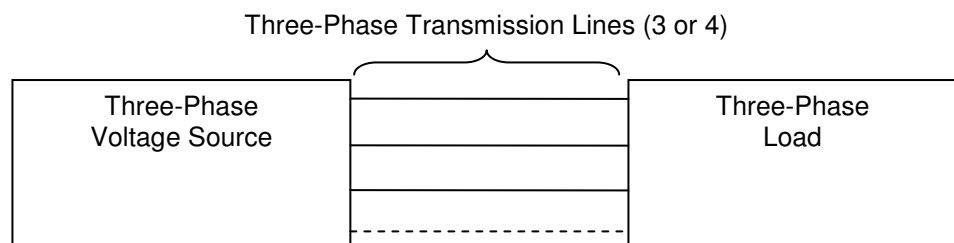
11.1 Introduction

Three-phase electric power is commonly used in larger electricity generation, transmission and distribution operations. Three-phase is a more economical way to transmit power since it uses less transmission line material than other systems at a given voltage. Although most households in US use single-phase power, generally transmission of power is done using three-phase power.

Additionally, three-phase power is used in industry where there is a need for large-load motors that require near constant torque. The instant power of each phase peaks at a different time results in a more constant power delivery which results in constant torque and reduces vibration in machinery.

Full coverage of three-phase circuit is the focus of Electrical Engineers specializing in Power and is beyond the scope of this text. In this chapter, we will focus on the balance three-phase power in steady-state sinusoidal condition which provides an excellent introduction to three-phase power systems.

In a three-phase system, three wires carry the three alternating currents and it may or may not have a neutral wire. In typical transmission systems there are step-up and step-down transformers in order to increase and decrease the voltage levels. High voltage transmission lines outside of populated areas may carry as high as one million volts and have to be stepped-down in multiple stages to 110 volts used in typical US House Hold. In this chapter, we will be working with three-phase system models that directly connect between the generation (Source) and consumption (Load) facilities as shown below:

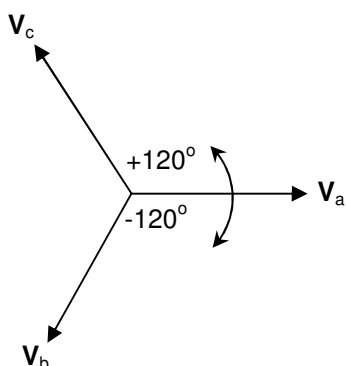


11.2. Balanced Three-Phase Voltages

The first step is to define balanced three-phase Voltages. Balanced three-phase voltages refer to three sinusoidal voltages with the same amplitude and same frequency that are exactly 120° out of phase with each other. a, b and c are used to name the phases as listed here:

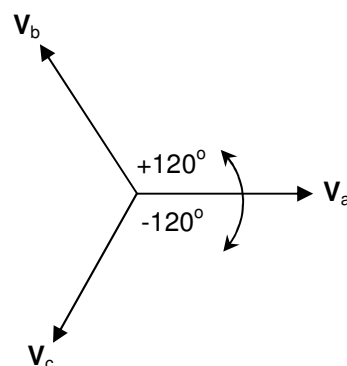
- a-phase voltage
- b-phase voltage
- c-phase voltage

The phase voltage sequencing is determined by the system designer. The two common sequences are abc (Positive) phase sequence and acb (Negative) phase sequence as shown below:



abc (or Positive) Phase Sequence

$$\begin{aligned} V_a &= V_m \angle 0^\circ \\ V_b &= V_m \angle -120^\circ \quad \text{Lags a by } 120^\circ \\ V_c &= V_m \angle +120^\circ \quad \text{Leads a by } 120^\circ \end{aligned}$$



acb (or Negative) Phase Sequence

$$\begin{aligned} V_a &= V_m \angle 0^\circ \\ V_b &= V_m \angle +120^\circ \quad \text{Leads a by } 120^\circ \\ V_c &= V_m \angle -120^\circ \quad \text{Lags a by } 120^\circ \end{aligned}$$

Notes: Counter Clockwise is the positive direction to read angles

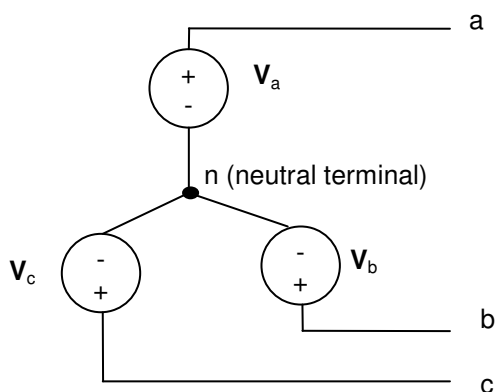
The balanced three-phase voltages have three important characteristics which are listed below:

1. Sum of phase voltages is zero $\rightarrow V_a + V_b + V_c = 0$
2. Sum of instantaneous voltages is zero $\rightarrow v_a + v_b + v_c = 0$
3. Once you know one of the voltages in a balanced three-phase voltage, you know them all since they have identical frequency and amplitude. The only difference is that each voltage is out of phase with the other by 120° .

11.3. Three-Phase Voltage Sources

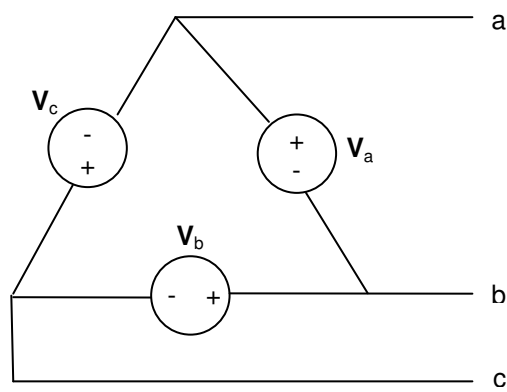
Three Phase Voltage Source or generator configuration is typically made up of a three set of winding equally space around a permanent magnet. As the magnet rotates, each winding generates one of the phase voltages. As discussed earlier in the case of balanced three-phase system, voltages will have the same magnitude but 120° phase difference.

The three windings may be interconnected either in a Wye(Y) or Delta(Δ) configuration as shown below:



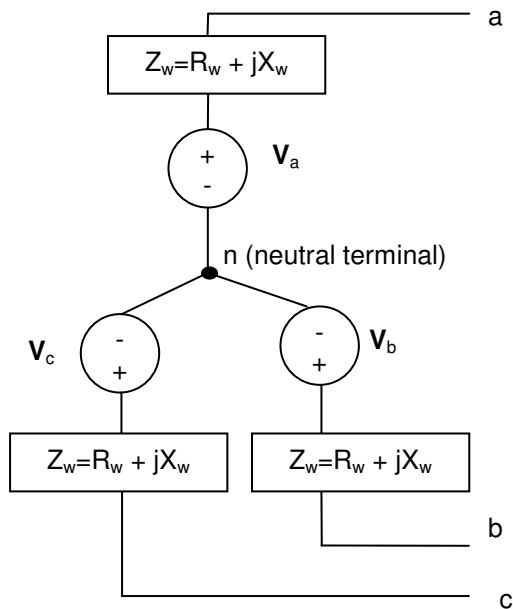
Y-Connected Source (ideal)

“n” terminal is referred to as the neutral terminal and may not have an external connection.

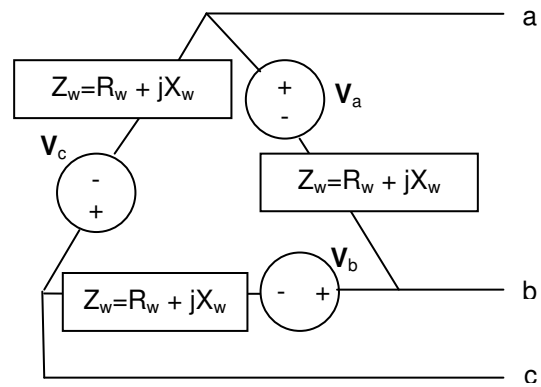


Δ -Connected Source (ideal)

The above diagram assumed that the generator windings are ideal with no Impedance. Although designers make every effort to minimize the winding impedance, sources will have some amount of impedance inherent in their construction which will effect the system operation. Therefore, the more accurate model of a three-phase generator is to include the impedance for each phase. R_w and X_w represent the winding resistance and reactance respectively. Since this is a balanced system, all three windings have the same impedance. The following diagram shows the balanced three-phase source model with the impedance:

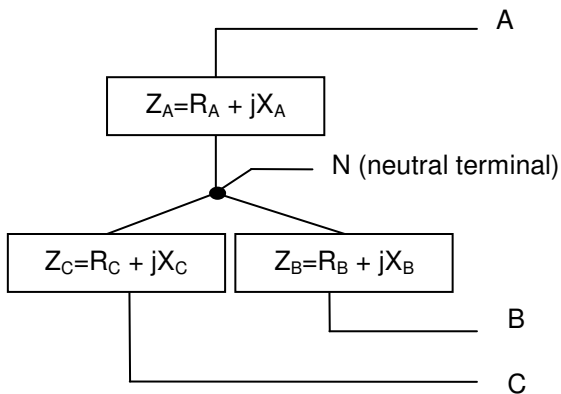


Y-Connected Source



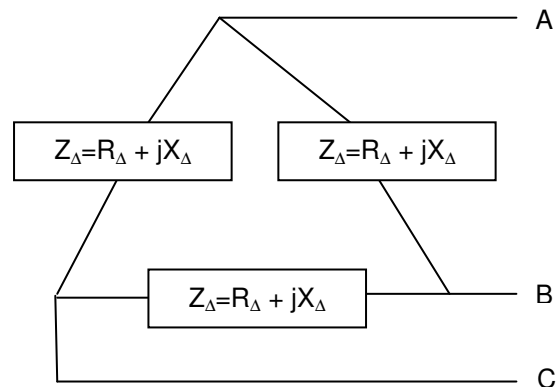
Δ-Connected Source

Although, the three-phase power generated may be converted to other configurations such as 2-phase and single phase, we will focus on its use on three-phase load. Three-phase load similar to three-phase source may be either configured as a Wye(Y) or Delta(Δ). Here are the typical load configurations:



Y-Connected Load

"n" terminal is referred to as the neutral terminal and may not have an external connection.

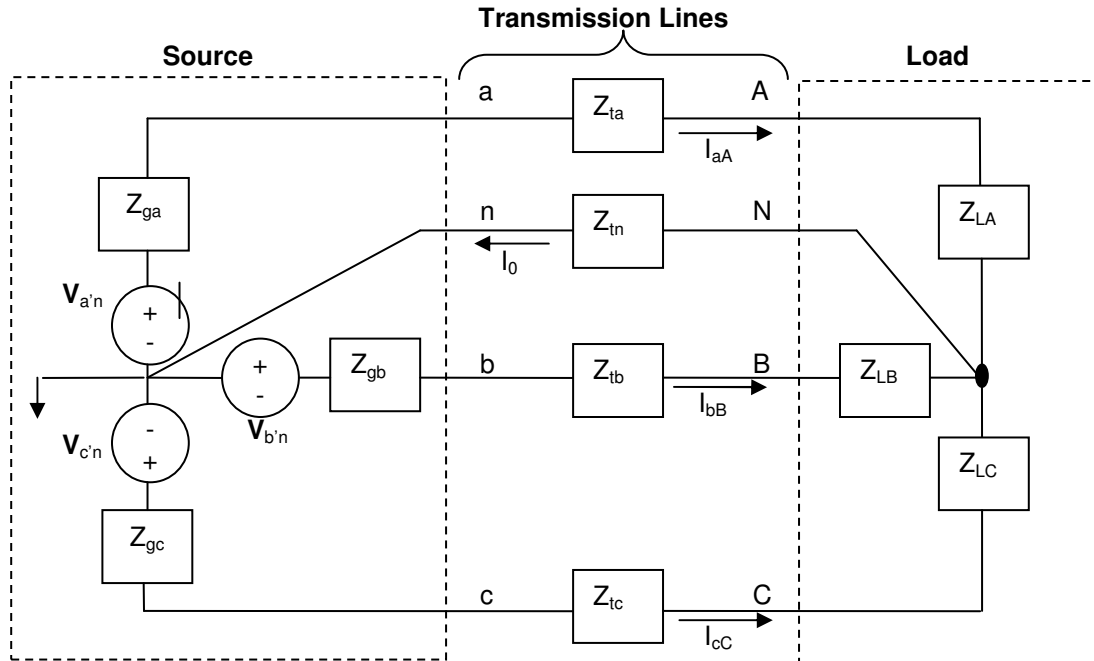


Δ-Connected Load

Since there are two source configurations and similarly two configuration of load, we have 4 possible way of connecting load and source ($Y \rightarrow Y$, $Y \rightarrow \Delta$, $\Delta \rightarrow Y$, $\Delta \rightarrow \Delta$). In the remainder of this section, two of the four possible configurations are discussed.

❖ Analysis of **Y→Y** Connection

Here, we will analyze the circuit resulting from connecting a Y-configured source to a Y-configured load. In the Y-Y circuit, in addition to the three transmission lines, it includes a neutral (4th connection transmission line). Furthermore, the transmission lines are modeled by their impedances (Z_{ta} , Z_{tb} , Z_{tc} , Z_{tn}). Below is the resulting circuit:



A single node voltage equation can be used to describe the circuit.

$$\frac{V_N}{Z_{tn}} + \frac{V_N - V_{a'n}}{Z_{La} + Z_{ta} + Z_{ga}} + \frac{V_N - V_{b'n}}{Z_{Lb} + Z_{tb} + Z_{gb}} + \frac{V_N - V_{c'n}}{Z_{Lc} + Z_{tc} + Z_{gc}} = 0$$

We can Simplify this equation by using the conditions for a balanced three phase circuit:

- | | | |
|--|---|-----------------------------------|
| (1) The Voltage sources are balanced | → | $V_{a'n} + V_{b'n} + V_{c'n} = 0$ |
| (2) Internal impedances of each source are identical | → | $Z_{ga} = Z_{gb} = Z_{gc}$ |
| (3) Impedances of each transmission are identical | → | $Z_{ta} = Z_{tb} = Z_{tc}$ |
| (4) Impedances of each load are identical | → | $Z_{La} = Z_{Lb} = Z_{Lc}$ |

Letting $Z_\phi = Z_{La} + Z_{ta} + Z_{ga} = Z_{Lb} + Z_{tb} + Z_{gb} = Z_{Lc} + Z_{tc} + Z_{gc}$ which lead to the node voltage equation to be rewritten as:

$$\frac{V_N}{Z_{tn}} + \frac{3V_N - (V_{a'n} + V_{b'n} + V_{c'n})}{Z_\phi} = 0$$

First condition of balanced 3-phase says $V_{a'n} + V_{b'n} + V_{c'n} = 0$ therefore:

$$\frac{V_N}{Z_{tn}} + \frac{3V_N}{Z_\phi} = 0 \Rightarrow V_N \left(\frac{1}{Z_0} + \frac{3}{Z_\phi} \right) = 0$$

Given that Impedances is not zero → **$V_N = 0$**

This means there is no difference between source and load neutral $\rightarrow \mathbf{I_0 = 0}$. Therefore, in a balanced three-phase Y \rightarrow Y connected circuit, the designer may either remove the neutral conductor or replaced it by short when modeling balanced three-phase circuits.

Next step is to write the line currents equation as shown below:

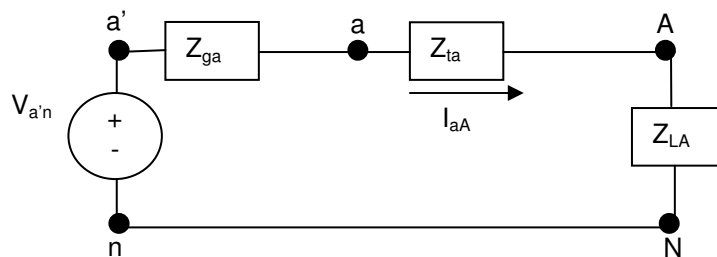
$$I_{aA} = \frac{V_{an} - V_N}{Z_\phi}; \quad I_{bB} = \frac{V_{bn} - V_N}{Z_\phi}; \quad I_{cC} = \frac{V_{cn} - V_N}{Z_\phi};$$

where $Z_\phi = Z_A + Z_{1a} + Z_{ga} = Z_B + Z_{1b} + Z_{gb} = Z_C + Z_{1c} + Z_{gc}$

The three currents similar to their respective voltages have the same magnitude and frequency, and are 120° out of phase.

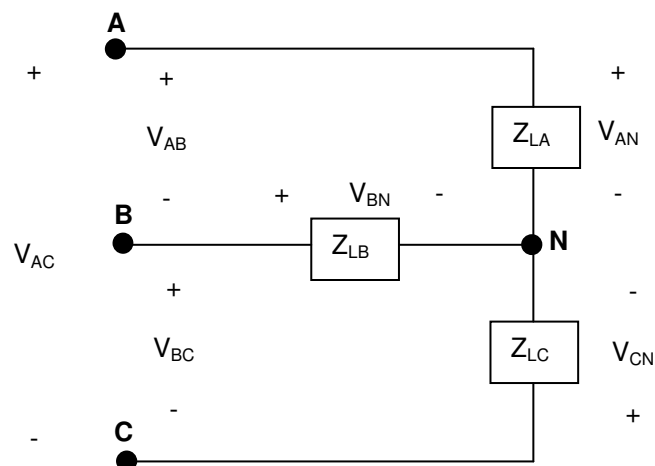
Therefore, it is understood that we only need to find voltage and current for one phase and then shift them by 120° to find the voltage and current for other phases. In other word, we can solve a single phase equivalent circuit and extend the result to other phases.

Here is a single-phase equivalent circuit for the Y \rightarrow Y circuit:



Note: the neutral current here has only one component so it is not equal to three-phase $I_0 = I_{aA} + I_{bB} + I_{cC}$

The above circuit may be used to find I_{aA} by applying KVL. Now that we have I_{aA} , we can use the following equivalent diagram of loads to calculate load voltages:



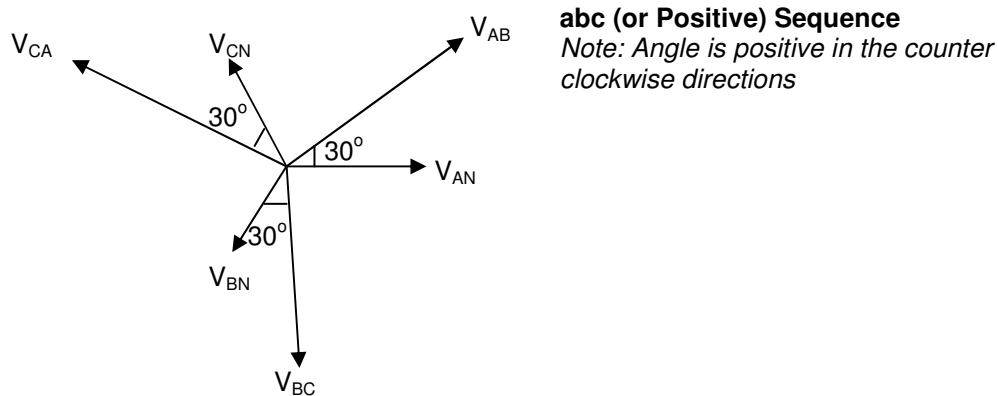
Based on I_{aA} from single phase circuit, we calculate magnitude of Line-to-Neutral Voltages ($V_\phi = I_{aA} Z_A$) for abc (positive) configuration, Phase voltages may be written as:

$$\begin{aligned} V_{AN} &= V_\phi \angle 0^\circ \\ V_{BN} &= V_\phi \angle -120^\circ \\ V_{CN} &= V_\phi \angle +120^\circ \end{aligned}$$

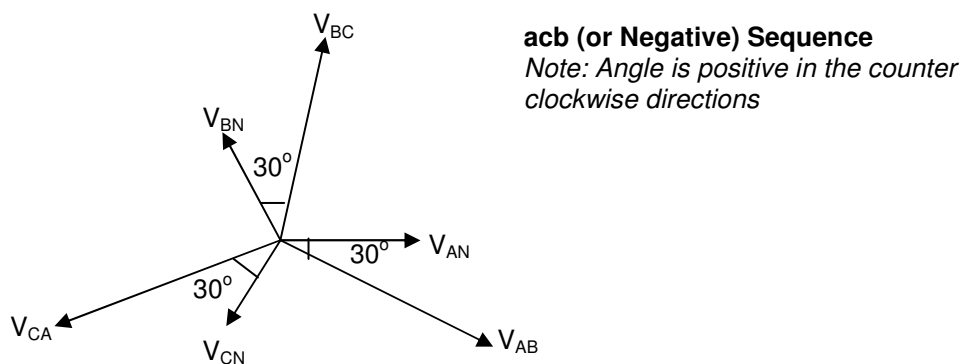
Line-to-Line voltages can be found using the KVL as shown below:

$$\begin{aligned} V_{AB} &= V_{AN} - V_{BN} = V_\phi \angle 0^\circ - V_\phi \angle -120^\circ = \sqrt{3} V_\phi \angle 30^\circ = \sqrt{3} V_{AN} \angle 30^\circ \\ V_{BC} &= V_{BN} - V_{CN} = V_\phi \angle -120^\circ - V_\phi \angle +120^\circ = \sqrt{3} V_\phi \angle -90^\circ = \sqrt{3} V_{BN} \angle 30^\circ \\ V_{CA} &= V_{CN} - V_{AN} = V_\phi \angle +120^\circ - V_\phi \angle 0^\circ = \sqrt{3} V_\phi \angle +150^\circ = \sqrt{3} V_{CN} \angle 30^\circ \end{aligned}$$

Phasor diagram of Line-to-line and line-to-neutral voltages for abc or Positive Sequence:



One could also go through the same process too draw the phase diagram for acb or negative sequence. The only difference would that the line-to-line voltage would lag the line-to-neutral voltage as shown below:



Finally, here are some of the terminologies which are common to this field:

- Line Voltage refers to voltage across a pair of line or line-to-line voltage (ie. V_{BC})
- Phase Voltage refers to voltage across a single phase or line-to-neutral voltage ((ie. V_{BN}))

- Line current refers to the current in a single line
- Phase current refers to current in a single phase
- Since three-phase is used for power, effective or rms value is commonly used. So the statement “transmission line is rated for 500 kV”, means that the nominal value of the rms line-to-line voltage is 500,000 V.
- The Greek letter Φ is used to refer to per-phase quantities: I_Φ (Current/ Phase), V_Φ (Voltage/ Phase), Z_Φ (Impedance/ Phase), P_Φ (Real or Average Power/ Phase), Q_Φ (Reactive power/ Phase)
- Δ and Y Connection Comparison
 - (1) Phase and Line voltage are the same in Δ Connection
 - (2) Phase and line current are the same in Y connection.

➤ Example – **Y→Y** Balanced Three-phase Circuit Analysis

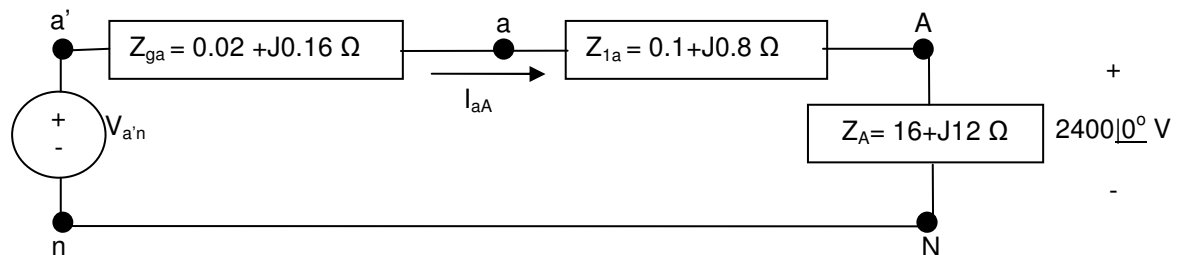
The phase voltage at the terminal of a balanced three-phase Y connected load is 2400 V. The load has an impedance of $(16 + j12) \Omega/\Phi$ and is fed by a line having an impedance of $0.1 + j0.8 \Omega/\Phi$. The Y-connected source at the sending end of the line has a phase sequence of acb and an internal impedance of $(0.02 + j0.16) \Omega/\Phi$. Use the a-phase voltage at the load as the reference to calculate:

- the line currents I_{aA} , I_{bB} and I_{cC} .
- the line voltages at the source, V_{ab} , V_{bc} and V_{ca}
- the internal phase to neutral voltages at the source, $V_{a'n}$, $V_{b'n}$ and $V_{c'n}$.

Solution:

a) Line currents?

Analyze the single phase circuit for a-phase



$$\text{KVL around the loop} \rightarrow I_{aA} = \frac{2400}{(16 + j12)} = \frac{2400}{20 \angle 36.87^\circ} = 120 \angle -36.87^\circ \text{ A}$$

Using acb or negative sequence to write the line currents therefore:

$$I_{aA} = I_m \angle \theta^\circ = 120 \angle -36.87^\circ \text{ A}$$

$$I_{bB} = I_m \angle \underline{120 + \theta^\circ} = 120 \angle +83.13^\circ \text{ A}$$

$$I_{cC} = I_m \angle \underline{-120 + \theta^\circ} = 120 \angle -156.87^\circ \text{ A}$$

b) Line Voltages?

$$V_{AN} = 2400 \angle 0^\circ V$$

$$V_{an} = (I_m \angle \theta^\circ)(Z) = (120 \angle -36.87^\circ)(16.1 + j12.8) = 2468.18 \angle 1.62^\circ V$$

$$V_{ab} = (\sqrt{3} \angle -30^\circ)(2468.18 \angle 1.62^\circ) = 4275.02 \angle -28.38^\circ V$$

Using the Blance three – phased circuit conditions (Shift by 120°) – acb sequence

$$V_{bc} = 4275.02 \angle 120 - 28.38^\circ = 4275.02 \angle 91.62^\circ V$$

$$V_{ca} = 4275.02 \angle -120 - 28.38^\circ = 4275.02 \angle -148.38^\circ V$$

c) Internal Voltages

$$V_{a'n} = (I_m \angle \theta^\circ)(Z) = (120 \angle -36.87^\circ)(16.12 + j12.96) = 2482.05 \angle 1.93^\circ V$$

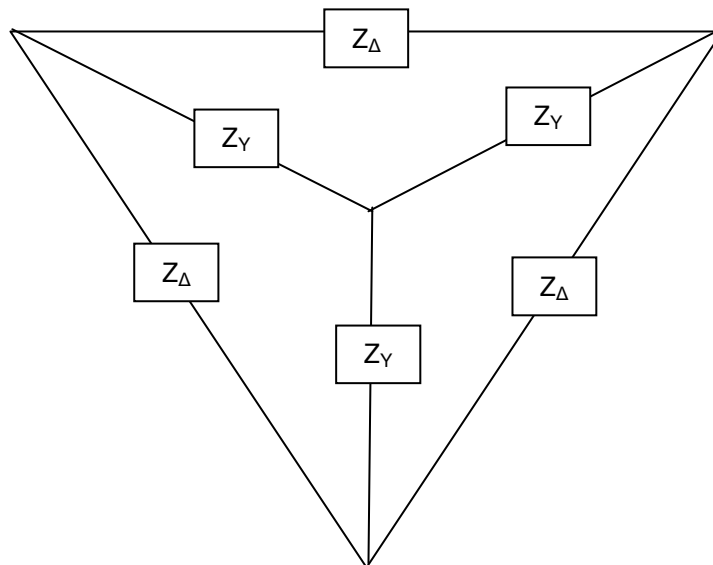
Using the Blance three – phased circuit conditions (Shift by 120°) – acb sequence

$$V_{b'n} = 2482.05 \angle 120 + 1.93^\circ = 2482.05 \angle 121.93^\circ V$$

$$V_{c'n} = 2482.05 \angle -120 + 1.93^\circ = 2482.05 \angle -118.07^\circ V$$

❖ Analysis of $Y \rightarrow \Delta$ Connection

In this section, we will analyze the circuits resulting from connecting a Y-configured source to a Δ -configured load. The simplest approach to analysis is to use the $\Delta \rightarrow Y$ impedance network transformation from the earlier chapter to convert the Δ load to Y Load as shown below:

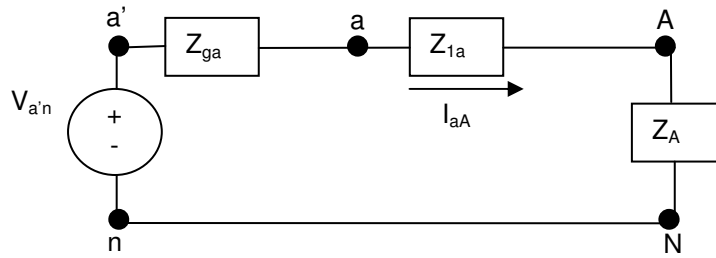


Delta \leftrightarrow Y Transformation
 $Z_\Delta = 3Z_Y$ or $Z_Y = Z_\Delta/3$

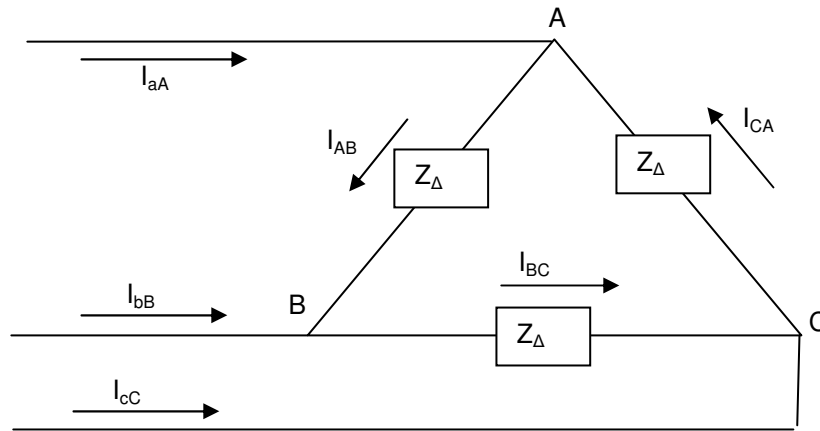
Note that in a Balanced three phase Δ load, all loads are equal which means its Y equivalent will also have equal loads.

Now that the load is in Y configuration, we can apply the learning from $Y \rightarrow Y$ analysis section as shown below:

Step 1) Using the above transformation we can now use the same techniques as Y-Y circuit by replacing Z_Δ with Z_Y . After the conversion, the following single phase equivalent circuit may be used for analysis.



Step 2) The following circuit can be used to establish relationship between line currents and phase current in a balanced Δ load.



(1) Phase Current (abc or positive sequence)

$$I_{AB} = I_{\phi} \angle 0^{\circ} ; I_{BC} = I_{\phi} \angle -120^{\circ} ; I_{CA} = I_{\phi} \angle 120^{\circ}$$

(Note: I_{AB} is arbitrary selected to be the reference and I_{ϕ} represents the magnitude of line current)

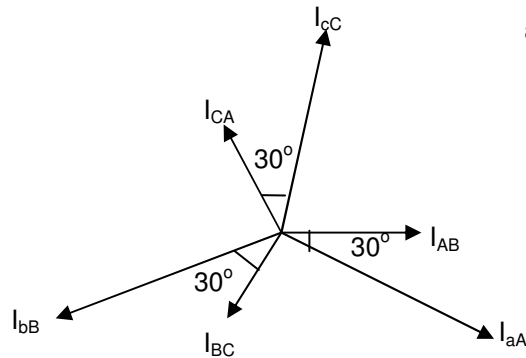
(2) Line Current (abc or positive sequence)

$$I_{aA} = I_{AB} - I_{CA} = I_{\phi} \angle 0^{\circ} - I_{\phi} \angle 120^{\circ} = \sqrt{3} I_{\phi} \angle -30^{\circ} = \sqrt{3} I_{AB} \angle -30^{\circ}$$

$$I_{bB} = I_{BC} - I_{AB} = I_{\phi} \angle -120^{\circ} - I_{\phi} \angle 0^{\circ} = \sqrt{3} I_{\phi} \angle -150^{\circ} = \sqrt{3} I_{BC} \angle -30^{\circ}$$

$$I_{cC} = I_{CA} - I_{BC} = I_{\phi} \angle 120^{\circ} - I_{\phi} \angle -120^{\circ} = \sqrt{3} I_{\phi} \angle 90^{\circ} = \sqrt{3} I_{CA} \angle -30^{\circ}$$

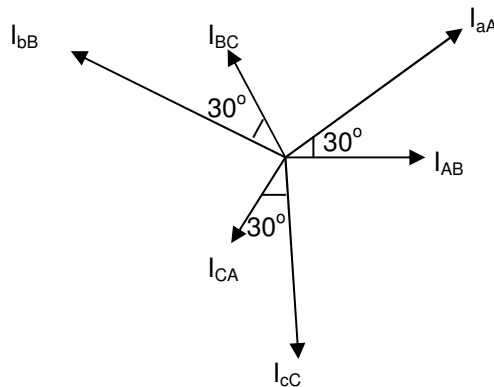
Phasor diagram of Line and phase currents



abc (or Positive) Sequence

Note: Angle is positive in the counter clockwise directions

One could also go through the process for acb (or negative) three phase system – the only difference is that the line current leads the phase current by 30° .



acb (or Negative) Sequence

Note: Angle is positive in the counter clockwise directions

➤ Example 1 – Y→Δ connected circuit

The phase current I_{CA} in a balance three-phase Δ-connected load is $12 \angle -25^\circ$ A. If the phase sequence is positive, what is the value of I_{cC} ?

Solution:

$$\text{Line Current} = I_{cC} = \sqrt{3} \angle -30^\circ I_{CA} = \sqrt{3} (12) \angle -25 - 30^\circ = 20.8 \angle -55^\circ \text{ A}$$

➤ Example 2 -- Y→Δ connected circuit

The line voltage V_{AB} at the terminals of a balanced three-phase Δ-connected load is $3210 \angle 0^\circ$ V. The line current I_{aA} is $75 \angle -15^\circ$ A. For this circuit

- calculate the per-phase impedance of the load using positive phase sequence.
- Repeat part (a) using negative phase sequence.

Solution:

a) Positive Sequence - Z_L

$$I_{aA} = \sqrt{3} \angle -30^\circ I_{AB} \rightarrow I_{AB} = (I_{aA} \angle 30^\circ) / \sqrt{3} = (75 \angle 15^\circ) / \sqrt{3} = 43.3 \angle 15^\circ \text{ A}$$

$$Z_{\Delta L} = (V_{AB} / I_{AB}) = (3210 \angle 0^\circ) / (43.3 \angle 15^\circ) = 74.1 \angle -15^\circ \Omega$$

b) Negative Sequence - Z_L

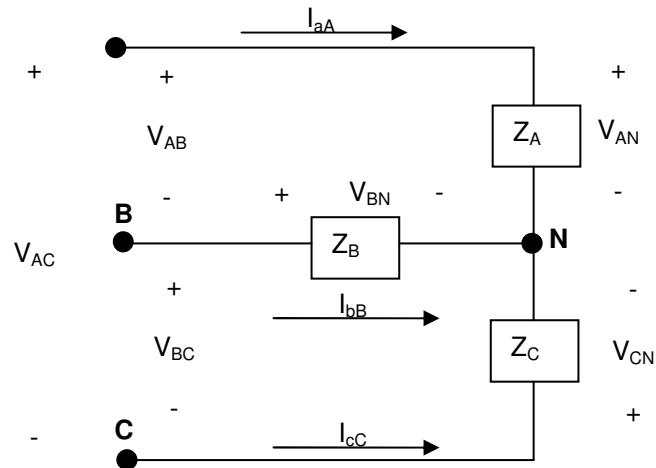
$$I_{aA} = \sqrt{3} \angle 30^\circ I_{AB} \rightarrow I_{AB} = (I_{aA} \angle -30^\circ) / \sqrt{3} = (75 \angle -45^\circ) / \sqrt{3} = 43.3 \angle -45^\circ \text{ A}$$

$$Z_{\Delta L} = (V_{AB} / I_{AB}) = (3210 \angle 0^\circ) / (43.3 \angle -45^\circ) = 74.1 \angle 45^\circ \Omega$$

❖ Summary of Y & Δ Loads

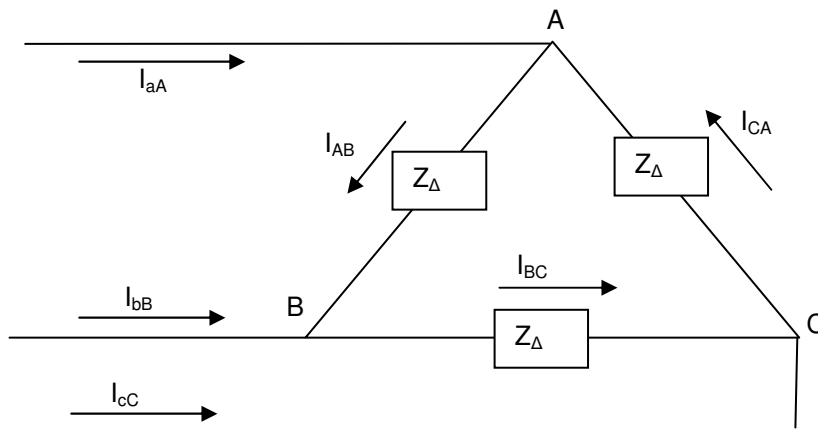
➤ Y Load

- Phase Voltage $\rightarrow V_{AN}, V_{BN}, V_{CN}$
- Line Voltage $\rightarrow V_{AB}, V_{AC}, V_{BC}$
- Phase Currents are the same as Line Currents $\rightarrow I_{aA}, I_{bB}, I_{cC}$



➤ Δ Load

- Line voltage and Phase Voltage are the same $\rightarrow V_{AB}, V_{AC}, V_{BC}$
- Phase Current $\rightarrow I_{AB}, I_{BC}, I_{CA}$
- Line Current $\rightarrow I_{aA}, I_{bB}, I_{cC}$



11.4. Balance Three-Phase Circuit Power Analysis

Power is an important parameter in circuit analysis since power represents circuit's ability to effect or sense the physical environment and other devices. Additionally, power is an integrative measure of current and voltage, the two main characteristics of any circuit. For these reasons, power calculation and analysis will be an important part of any type of circuit analysis and design.

At any point-of-time, circuit power is represented by instantaneous power, $p(t)$ or simply p . As discussed earlier Power delivered by a balanced three-phase circuit is constant with major benefit being the constant torque developed at the shaft of the three-phase motor is constant which leads to less vibration in machinery powered by three phase motors.

The following few paragraphs show an approach to prove that instantaneous power is constant in three-phase balanced circuits. Let's start by calculating per phase as shown below:

$$\begin{aligned} p_A &= v_{AN} i_{aA} = V_m I_m \cos(wt) \cos(wt - \theta_\phi) \\ p_B &= v_{BN} i_{bB} = V_m I_m \cos(wt - 120^\circ) \cos(wt - \theta_\phi - 120^\circ) \\ p_C &= v_{CN} i_{cC} = V_m I_m \cos(wt + 120^\circ) \cos(wt - \theta_\phi + 120^\circ) \end{aligned}$$

Where:

- * v_{AN} is voltage between a phase and neutral line
- * θ_ϕ is the power phase angle ($\theta_{vA} - \theta_{iA}$);
- * Assume positive phase sequence (abc)
- * V_m and I_m represent the maximum amplitude of the phase voltage of line current.
- * $V_m = \sqrt{2} V_\phi$ and $I_m = \sqrt{2} I_\phi$ relating maximum to the effective or rms value per phase.

The Total Instantaneous Power = $p_T = p_A + p_B + p_C = 1.5 V_m I_m \cos(\theta_\phi)$

Conclusion: Total Instantaneous Power (p_T) is constant with respect to time (t)."

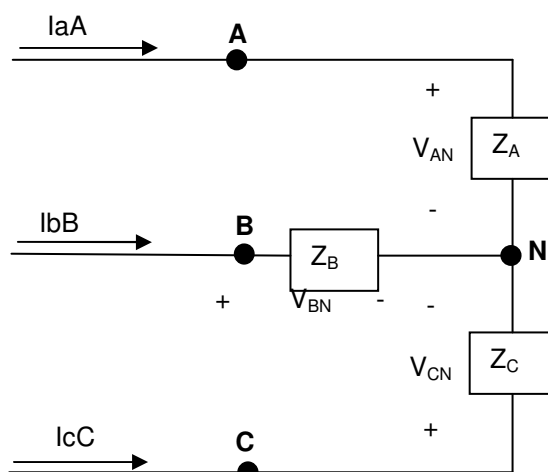
In the remainder of this section, we will be discussing average, reactive and complex power in a balanced three-phase circuit. We will start with a **Y**-load power analysis and follow it with a discussion of **Δ**-Load power.

❖ Power Calculation in a Three-Phase Balanced **Y**-load

Below is diagram of a typical **Y**-load with voltages and currents.

Notes

- V_{AB}, V_{BC}, V_{AC} are Line Voltages.
- V_{AN}, V_{BN}, V_{CN} are Phase Voltages.
- Line and Phase Currents are the same in Y Connection.



Apply the Average Power equations $\{P = |V| |I| \cos(\theta_v - \theta_i)\}$ to calculate Average Power for each of the three phases:

$$P_A = |V_{AN}| |I_{aA}| \cos(\theta_{vA} - \theta_{iA})$$

$$P_B = |V_{BN}| |I_{bB}| \cos(\theta_{vB} - \theta_{iB})$$

$$P_C = |V_{CN}| |I_{cC}| \cos(\theta_{vC} - \theta_{iC})$$

Note: all Voltage and Current are written in term of their rms or effective values.

The following conditions are true since the analysis is being done on a three-phase balanced circuit:

$$V_\phi = |V_{AN}| = |V_{BN}| = |V_{CN}|$$

$$I_\phi = |I_{aA}| = |I_{bB}| = |I_{cC}|$$

$$\theta_\phi = \theta_{vA} - \theta_{iA} = \theta_{vB} - \theta_{iB} = \theta_{vC} - \theta_{iC}$$

Therefore, per phase average power can be rewritten as:

$$\text{Average Power per phase} = P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos(\theta_\phi)$$

Total Average Power delivered to all three phases can be written as:

$$\text{Total Average Power} = P_T = P_A + P_B + P_C = 3P_\phi = 3 V_\phi I_\phi \cos(\theta_\phi)$$

Total Average Power can be written in term of line voltage and current:

$$\text{Total Average Power} = P_T = 3(V_L / \sqrt{3}) I_L \cos(\theta_\phi) = \sqrt{3} V_L I_L \cos(\theta_\phi) \quad \text{Where}$$

* V_L and I_L are the rms magnitudes of the line voltage and current.

* θ_ϕ is the phase angle difference between phase voltage and phase current

Using similar process, Reactive Power (Q) in a three-phase balance Y-load can be calculated by the following equation:

$$\text{Reactive Power per phase} = Q_\phi = V_\phi I_\phi \sin(\theta_\phi)$$

$$\text{Total Reactive Power} = Q_T = 3Q_\phi = \sqrt{3} V_L I_L \sin(\theta_\phi) \quad \text{Where}$$

* V_L and I_L are the rms magnitudes of the line voltage and current.

* θ_ϕ is the phase angle difference between phase voltage and phase current

Complex Power ($S = V I^*$) can be expressed in-term of load using the equations derived earlier for Reactive and Average Power:

$$\text{Complex power per phase} = S_\phi = V_{AN} I_{aA}^* = V_{BN} I_{bB}^* = V_{CN} I_{cC}^* = V_\phi I_\phi^* = P_\phi + jQ_\phi$$

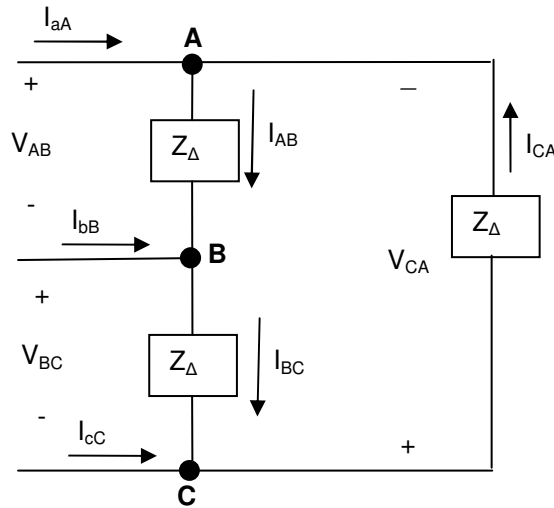
$$\text{Total Complex power} = S_T = 3S_\phi = \sqrt{3} V_L I_L \angle \theta_\phi$$

❖ Power Calculation in a Three-Phase Balanced Δ -load

Below is diagram of a typical Δ -load with phase voltages and line currents.

Notes:

- I_{aA} , I_{bB} , I_{cC} are Line Currents.
- I_{AB} , I_{BC} , I_{CA} are Phase Currents.
- Line and Phase Voltages are the same in Delta Connection



Apply the Average Power equations $\{P = |V| |I| \cos(\theta_v - \theta_i)\}$ to calculate Average Power for each of the three phases of a Three-Phase Balanced Δ -Load:

$$P_A = |V_{AB}| |I_{AB}| \cos(\theta_{V_{AB}} - \theta_{I_{AB}})$$

$$P_B = |V_{BC}| |I_{BC}| \cos(\theta_{V_{BC}} - \theta_{I_{BC}})$$

$$P_C = |V_{CA}| |I_{CA}| \cos(\theta_{V_{CA}} - \theta_{I_{CA}})$$

Note: all Voltage and Current are written in term of the rms values.

The following conditions are true since the analysis is being done on a three-phase balanced circuit:

$$V_\phi = |V_{AB}| = |V_{BC}| = |V_{CA}|$$

$$I_\phi = |I_{AB}| = |I_{BC}| = |I_{CA}|$$

$$\theta_\phi = \theta_{V_{AB}} - \theta_{I_{AB}} = \theta_{V_{BC}} - \theta_{I_{BC}} = \theta_{V_{CA}} - \theta_{I_{CA}}$$

Which means Average Power per phase = $P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos(\theta_\phi)$

Sum of powers for all phases or Total Average Power = $P_T = 3P_\phi = 3V_\phi I_\phi \cos(\theta_\phi)$

Total Average Power can be written in term of Load Current and Voltage:

$$\text{Total Average Power} = P_T = 3(I_L / \sqrt{3}) V_L \cos(\theta_\phi) = \sqrt{3} V_L I_L \cos(\theta_\phi) \quad \text{Where}$$

* V_L and I_L are the rms magnitudes of the line voltage and current.

* θ_ϕ is the phase angle difference between phase voltage and phase current

Using similar process, Reactive Power (Q) in a three-phase balance Δ -load can be calculated by the following equation:

$$\text{Reactive Power per phase} = Q_\phi = V_\phi I_\phi \sin(\theta_\phi)$$

$$\text{Total Reactive Power} = Q_T = 3Q_\phi = \sqrt{3} V_L I_L \sin(\theta_\phi) \quad \text{Where}$$

* V_L and I_L are the rms magnitudes of the line voltage and current.

* θ_ϕ is the phase angle difference between phase voltage and phase current

Complex Power ($S = V I^*$) can be expressed in-term of load using the equations derived earlier for Reactive and Average Power:

$$\text{Complex power per phase} = S_\phi = V_{AN} I_{aA}^* = V_{BN} I_{bB}^* = V_{CN} I_{cC}^* = V_\phi I_\phi^* = P_\phi + jQ_\phi$$

$$\text{Total Complex power} = S_T = 3S_\phi = \sqrt{3} V_L I_L \angle \theta_\phi$$

❖ Example - Power Calculation in a Balanced Three-phase Circuit

The complex power associated with each phase of a balanced load is $144 + j192$ kVA. The line voltage at terminals of the load is 2450 V. For this abc sequence circuit:

- What is the magnitude of the line current feeding the Y connected load?
- The load is delta connected and the impedance of each phase consists of a resistance in parallel with a reactance. Calculate R and X.
- The load is Wye connected, and the impedance of each phase consists of a resistance in series with a reactance. Calculate R and X.

Solution:

a) Magnitude of Line Current?

$$V_{AB} = \sqrt{3} V_{AN} \angle 30^\circ \rightarrow V_{AN} = (V_{AB} \angle -30^\circ) / \sqrt{3} = (2450 \angle -30^\circ) / \sqrt{3} = 1414.51 \angle -30^\circ \text{ V}$$

$$S_\phi = V_{AN} I_{aA}^* \rightarrow I_{aA}^* = S_\phi / V_{AN}$$

$$I_{aA}^* = 1000 (144 + j192) / (1414.51 \angle -30^\circ) = 1000(240 \angle 53.13^\circ) / (1414.51 \angle -30^\circ) = 169.67 \angle 83.13^\circ$$

$$|I_{aA}| = 169.67 \text{ Magnitude of the line current feeding the load}$$

b) R & X for Δ -Connected Load

$$S = V^2/Z^* \rightarrow Z^* = V^2/S = (2450)^2/(144000 + j192000) =$$

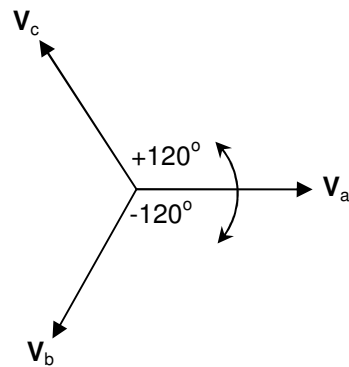
c) R & X for Y-Connected Load

$$Z_\phi = V_{AN} / I_{aA} = 1414.51 \angle -30^\circ / 169.67 \angle -83.13^\circ = 8.34 \angle 53.13^\circ = 5 + j 6.67 \Omega$$

$$\rightarrow R = 5 \Omega \quad \& \quad X = 6.67 \Omega$$

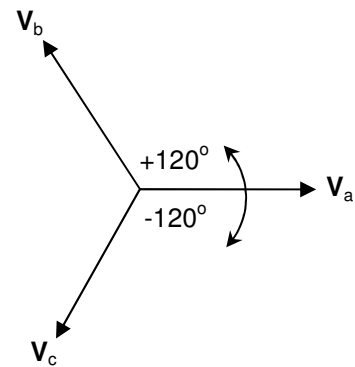
11.5. Summary

❖ Balanced 3-phase sequences



abc (or Positive) Phase Sequence

$$\begin{aligned} V_a &= V_m \angle 0^\circ \\ V_b &= V_m \angle -120^\circ \quad \text{Lags a by } 120^\circ \\ V_c &= V_m \angle +120^\circ \quad \text{Leads a by } 120^\circ \end{aligned}$$

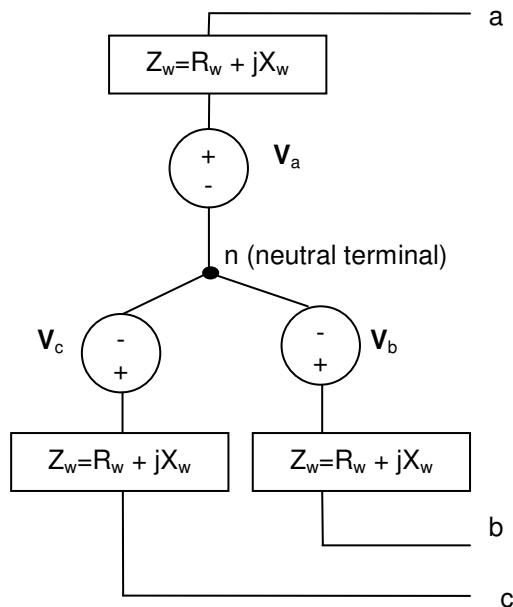


acb (or Negative) Phase Sequence

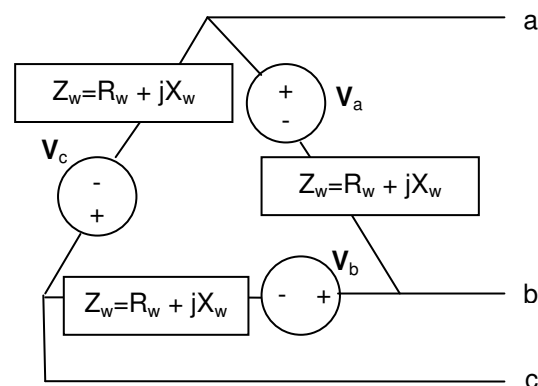
$$\begin{aligned} V_a &= V_m \angle 0^\circ \\ V_b &= V_m \angle +120^\circ \quad \text{Leads a by } 120^\circ \\ V_c &= V_m \angle -120^\circ \quad \text{Lags a by } 120^\circ \end{aligned}$$

Notes: Counter Clockwise is the positive direction to read angles

❖ 3-Phase Y and Δ Sources

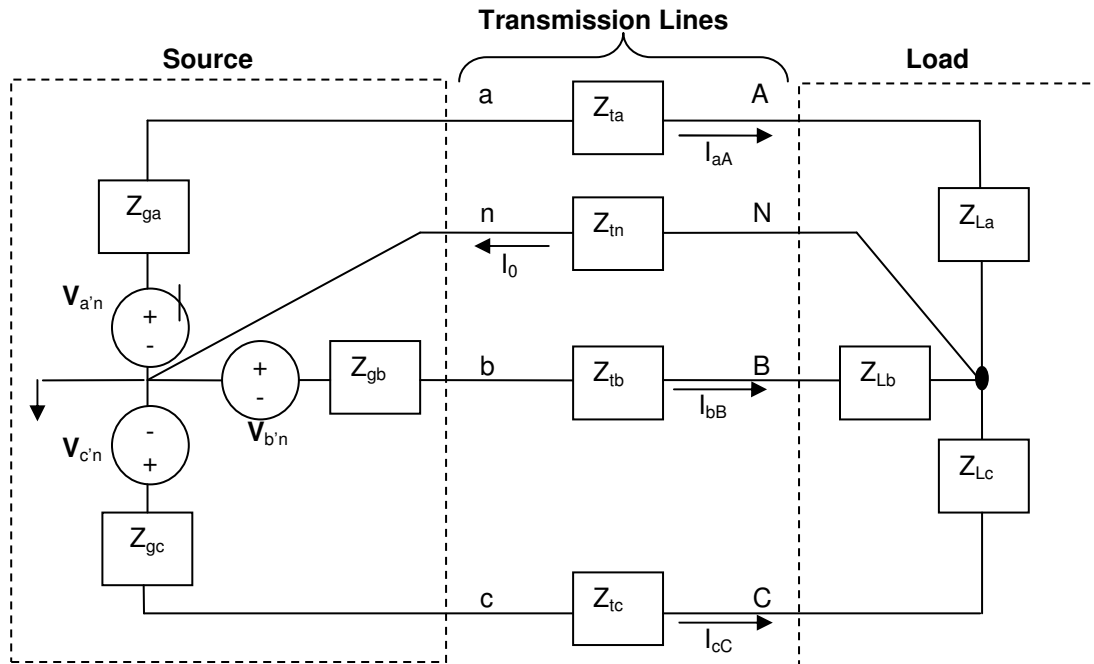


Y-Connected Source

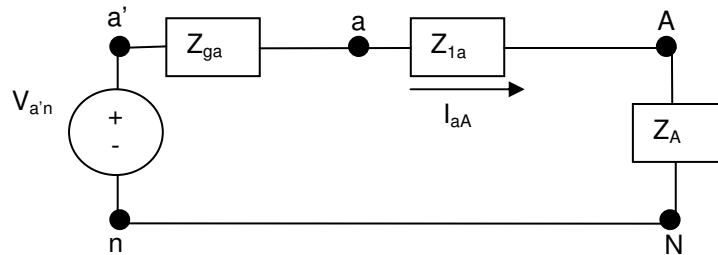


Δ -Connected Source

❖ Y→Y Connection



❖ Single Phase Analysis



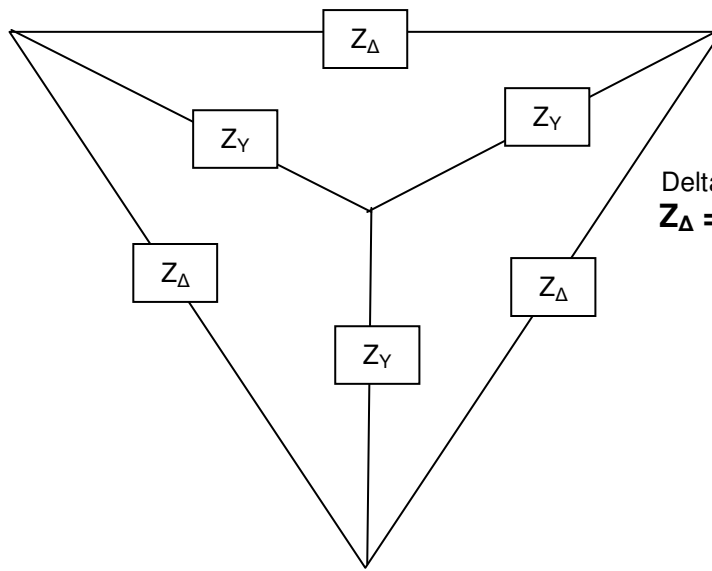
❖ Y-Connected Load

$$V_{AB} = V_{AN} - V_{BN} = V_{\phi} \angle 0^\circ - V_{\phi} \angle -120^\circ = \sqrt{3} V_{\phi} \angle 30^\circ = \sqrt{3} V_{AN} \angle 30^\circ$$

$$V_{BC} = V_{BN} - V_{CN} = V_{\phi} \angle -120^\circ - V_{\phi} \angle +120^\circ = \sqrt{3} V_{\phi} \angle -90^\circ = \sqrt{3} V_{BN} \angle 30^\circ$$

$$V_{CA} = V_{CN} - V_{AN} = V_{\phi} \angle 120^\circ - V_{\phi} \angle 0^\circ = \sqrt{3} V_{\phi} \angle +150^\circ = \sqrt{3} V_{CN} \angle 30^\circ$$

❖ **Y→Δ Connection**



Delta- \leftrightarrow Y Transformation
 $Z_{\Delta} = 3Z_Y$ or $Z_Y = Z_{\Delta}/3$

11.6. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 11. Another academic approach to the topic.
- ❖ en.Wikipedia.org, Search for “Three-phase electric Power”
An industrial view with animated graphics of the three-phase electric power operations.

11.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 12. Frequency Selective RLC Circuits (Filters)

Key Concepts and Overview

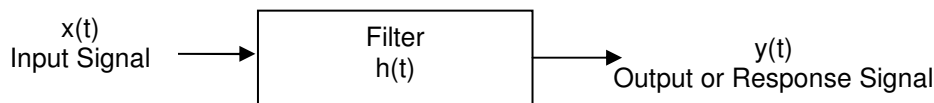
- ❖ Frequency Selective Circuits (Filters)
- ❖ First-order Low-Pass Filters
- ❖ First order High-Pass Filters
- ❖ Band Pass Filters
- ❖ Band Reject Filters
- ❖ Bode Plots
- ❖ Additional Resources

12.1. Frequency Selective Circuits (Filters)

Filters or frequency-selective circuits are used to amplify or attenuate signals based on the signal's frequency. Filters are used in all types of devices such as radios, televisions, audio/visual systems, medical diagnostics and cell phones.

One way to categorize filters is based on the elements used in their construction. Active Filters are designed using active elements such as Operational Amplifiers and are covered in the next chapter. This chapter focuses on Passive Filters which are designed only using passive circuit elements (R, L, and C).

As discussed earlier, any system can be defined by its input $x(t)$, output $y(t)$ and impulse function $h(t)$. $x(t)$ and $y(t)$ represent the signal and may represent either current or voltage. Basically, Impulse function relates the input to the output function in time domain as shown below:



$h(t)$ is referred to as the system impulse function response.

By leveraging the learning from the Phasor or frequency domain and allowing for $s=j\omega$, the system relationship may be re-drawn in Phasor or frequency domain as shown below:



$H(s)$ is the system transfer function response.

This system definition is also referred to as the s-domain. The key benefit of using the frequency domain is the introduction of transfer function $H(s)$ which relates the input to output with a simple algebra:

$$\text{System Transfer Function } H(s) = \frac{Y(s)}{X(s)} = \frac{\text{Output}}{\text{Input}} \text{ or } Y(s) = H(s)X(s) \text{ where } s=j\omega$$

In other word:

$$\text{System Transfer Function } H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \text{ or } Y(j\omega) = H(j\omega)X(j\omega)$$

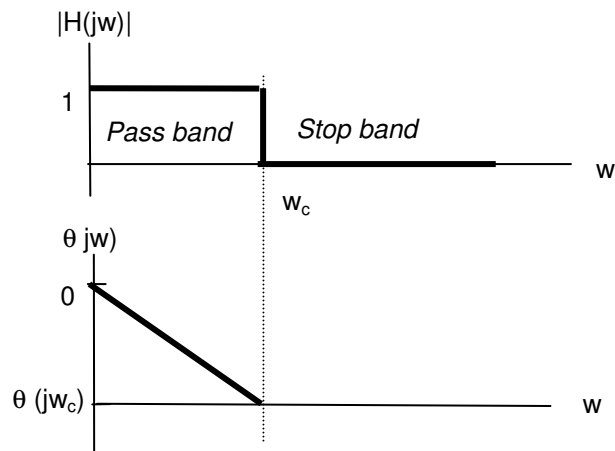
Transfer function, $H(j\omega)$, describes the relationship between input and output at a given frequency. $H(j\omega)$ serves a very important role in the discussion on filters design and circuit performance. For a complete understanding of the system and its transfer function with respect to frequency, we need to utilize both the magnitude of $H(j\omega)$ represented by $|H(j\omega)|$ and phase of $H(j\omega)$ represented by $\theta(j\omega)$. These two parameters completely describe the relationship between system input, $X(j\omega)$ and output, $Y(j\omega)$.

Filters are functionally categorized based on transfer function magnitude $|H(j\omega)|$ vs. frequency ω , and transfer function phase $\theta(j\omega)$ vs. frequency ω . The following diagrams show the transfer function magnitude and phase graphs for ideal low pass, high pass, band reject and band pass filters. It is important to remember that real filter's transfer function changes over time but here we will assume ideal

filter where the transfer function changes instantaneously.

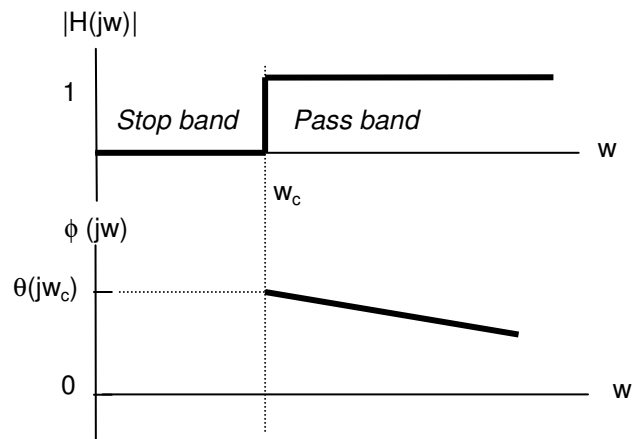
❖ Ideal Low Pass Filter

Low pass filter allows for signals with frequency from 0 to cut off frequency ω_c to pass through while blocking signals above cut off frequency ω_c .



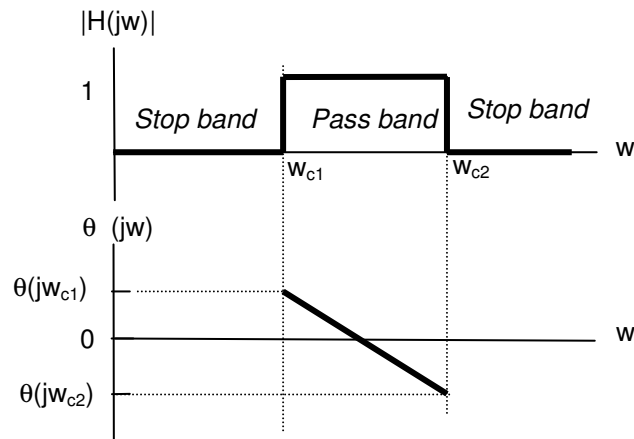
❖ Ideal High Pass Filter

High pass filter blocks signals with frequency from 0 to cut off frequency ω_c and passes through signals with frequency above cut off frequency ω_c .



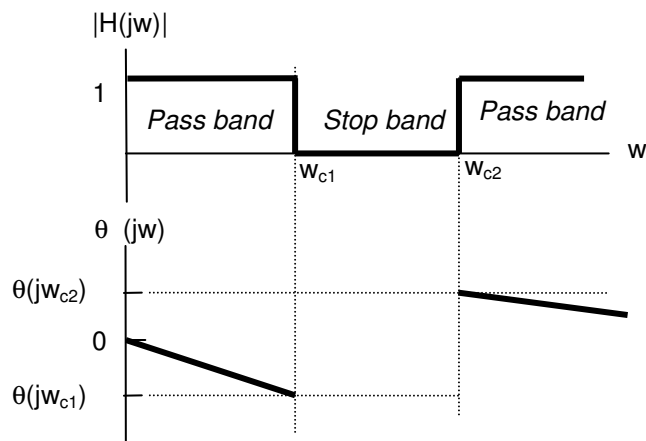
❖ Ideal Band Pass Filter

Band pass filter allows for signals with frequency between frequencies ω_{c1} and ω_{c2} to pass through while blocking signals with frequency outside the range.



❖ Ideal Band Reject Filter

Band reject filter blocks signals with frequency between frequencies ω_{c1} and ω_{c2} while allowing signals with frequency outside the range to pass through.



As discussed earlier, real filter transfer function does not change instantaneously from one level to another, therefore, it becomes important to define a criterion to calculate the cut off frequency (ω_c). The accepted definition is that the cut off frequency (ω_c) is the frequency at which half of the maximum power is being passed through the filter. Remember that $H(j\omega)$ represents the voltage transfer, therefore, the power transfer function which is the ratio of power would be square of $H(j\omega)$ which is the ratio of voltages.

$$|H(j\omega_c)|^2 = \frac{H_{\max}^2}{2} \quad \text{or} \quad |P(j\omega_c)| = \frac{P_{\max}}{2}$$

or

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$$

In the remaining sections of this chapter, typical circuits for each of the filter types will be presented along with their transfer functions in-terms of its components (R,L,and C). The transfer function is derived by application of node-voltage or mesh-current along with the transfer function definition as shown below:

$$|H(s)| = \left| \frac{V_o(s)}{V_i(s)} \right|$$

All passive filters are designed using passive components (R, L, and C). Further, they are analyzed without a load. As it is true in all passive circuits, load has the potential to significantly change the circuit characteristics. Therefore, it is important to design the filter circuit for a specific range of load.

Each of the following sections cover one type of filter as defined by the range of frequencies which signals are stopped and allowed to pass through.

12.2. First-order Low-Pass Filters

A low pass filter can be implemented by using a RC or RL circuit as shown here:

❖ Low Pass Series RL Circuit

This section analyzes the low pass series RL design and derives corresponding transfer function and cut off frequency.

step 1) Apply KVL to derive :

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{R}{s + \frac{R}{L}}}{\frac{R}{s + \frac{R}{L}}}$$

step 2) $H(j\omega) = \frac{\frac{R}{L}}{j\omega + \frac{R}{L}}$

$$|H(j\omega)| = \frac{\frac{R}{L}}{\sqrt{\omega^2 + (\frac{R}{L})^2}}$$

$$\theta(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

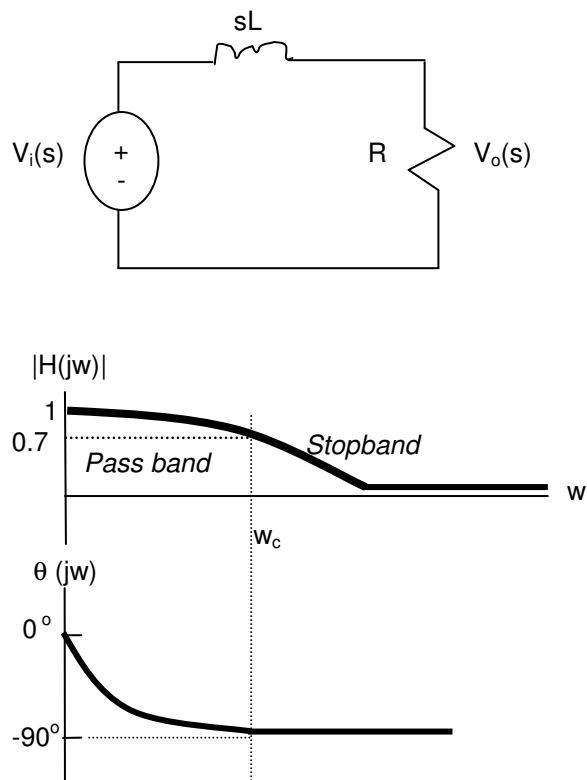
at $\omega_c \rightarrow |H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |1| = \frac{\frac{R}{L}}{\sqrt{\omega_c^2 + (\frac{R}{L})^2}}$$

$$\omega_c = \frac{R}{L}$$

implementation Independent $H(s)$:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$



➤ Example

Design a RL low pass filter with cut off frequency of 2,000 Hz and pass band gain of 1.

Solution:

❖ Low-Pass Series RC Circuit

This section analyzes the low pass series RC design and derive corresponding transfer function and cut off frequency.

step 1) Apply KVL to derive :

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

step 2) $H(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}$$

$$\theta(j\omega) = -\tan^{-1}(\omega RC)$$

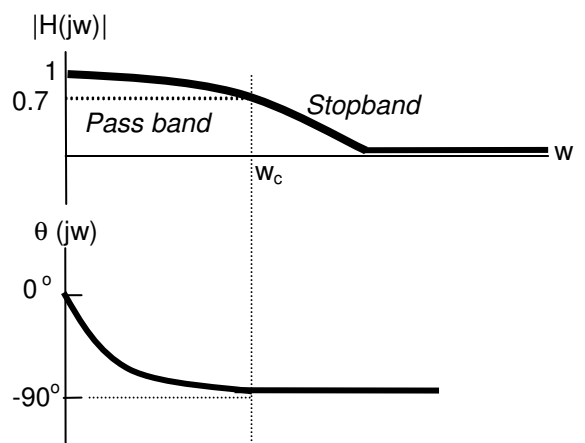
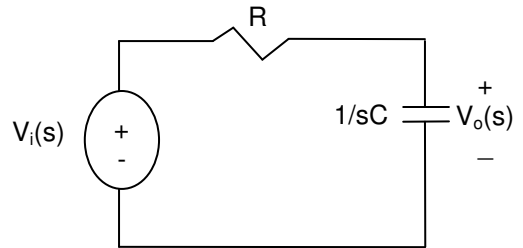
at $\omega_c \rightarrow |H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |1| = \frac{\frac{1}{RC}}{\sqrt{\omega_c^2 + \left(\frac{1}{RC}\right)^2}}$$

$$\omega_c = \frac{1}{RC}$$

implementation independent $H(s)$:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$



➤ Example

Design a RC low pass filter with cut off frequency of 80,000 radian/sec and pass band gain of 1.

Solution:

12.3. First order High-Pass Filters

A high pass filter can be implemented by using a series RC or RL circuit as shown here:

❖ High-Pass Series RL Circuit

This section analyzes the high pass series RL design and derives corresponding transfer function and cut off frequency.

step 1) Apply KVL to derive :

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s}{s + \frac{R}{L}}$$

step 2) $H(j\omega) = \frac{j\omega}{j\omega + \frac{R}{L}}$

$$|H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (\frac{R}{L})^2}}$$

$$\theta(j\omega) = 90^\circ - \tan^{-1}\left(\frac{\omega L}{R}\right)$$

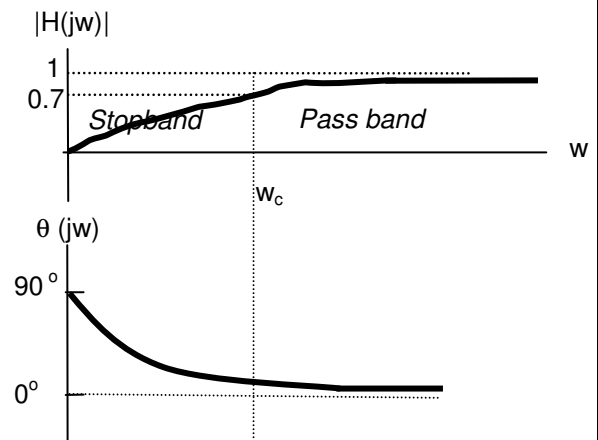
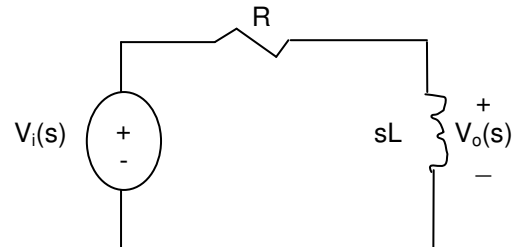
at $\omega_c \rightarrow |H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |1| = \frac{\omega_c}{\sqrt{\omega_c^2 + (\frac{R}{L})^2}}$$

$$\omega_c = \frac{R}{L}$$

implementation Independent $H(s)$:

$$H(s) = \frac{s}{s + \omega_c}$$



Note: Cut off frequency, $\omega_c = R/L = 1/\tau$ where τ = time Constant

➤ Example

Using resistors, inductors and a white noise generator (outputs full spectrum of 0 to infinity HZ), design a system that will drive away bats (Bats hear 20 kHz +/- 50 kHz). The system should not bother humans (hear 20 kHz max.) and dogs (hear 40 kHz max.)

Solution:

❖ High-Pass Series RC

This section analyzes the high pass series RC design and derives corresponding transfer function and cut off frequency.

step 1) Apply KVL to derive :

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s}{s + \frac{1}{RC}}$$

step 2) $H(jw) = \frac{jw}{jw + \frac{1}{RC}}$

$$|H(jw)| = \frac{w}{\sqrt{w^2 + (\frac{1}{RC})^2}}$$

$$\theta(jw) = 90^\circ - \tan^{-1}(wRC)$$

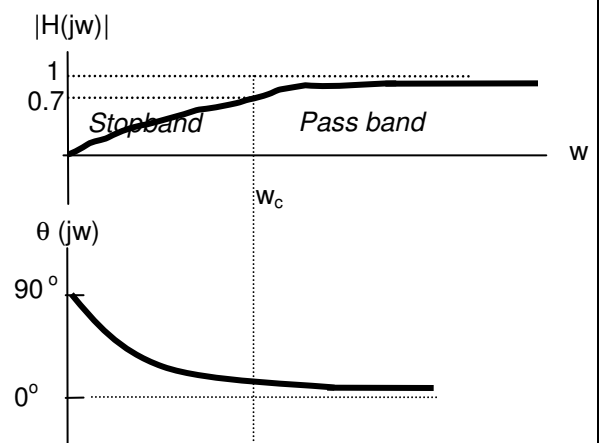
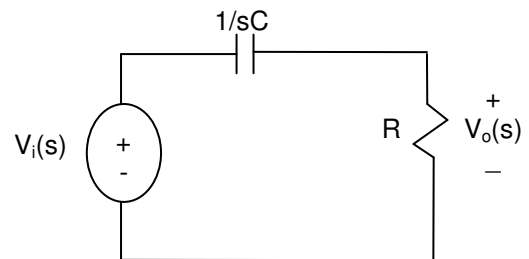
at $w_c \rightarrow |H(jw_c)| = \frac{1}{\sqrt{2}} H_{\max}$

$$|H(jw_c)| = \frac{1}{\sqrt{2}} |1| = \frac{w_c}{\sqrt{w_c^2 + (\frac{1}{RC})^2}}$$

$$w_c = \frac{1}{RC}$$

implementation Independent $H(s)$:

$$H(s) = \frac{s}{s + w_c}$$



Note: Cut off frequency, $w_c = 1/RC = 1/\tau$ where τ = time Constant

12.4. Band Pass Filters

Band pass filters as discussed earlier allow for signals with frequencies in the range between the cut off frequencies ω_{c1} and ω_{c2} to pass through. All other signals will be blocked. The two cut off frequencies

ω_{c1} and ω_{c2} occur when $|H(j\omega_c)| = (\frac{1}{\sqrt{2}})H_{\max}$.

Furthermore, we need to define resonant or center frequency (ω_0) which is the frequency where circuit's transfer function ($H(j\omega_c)$) is purely real. Mathematically ω_0 is the geometric center of the pass band

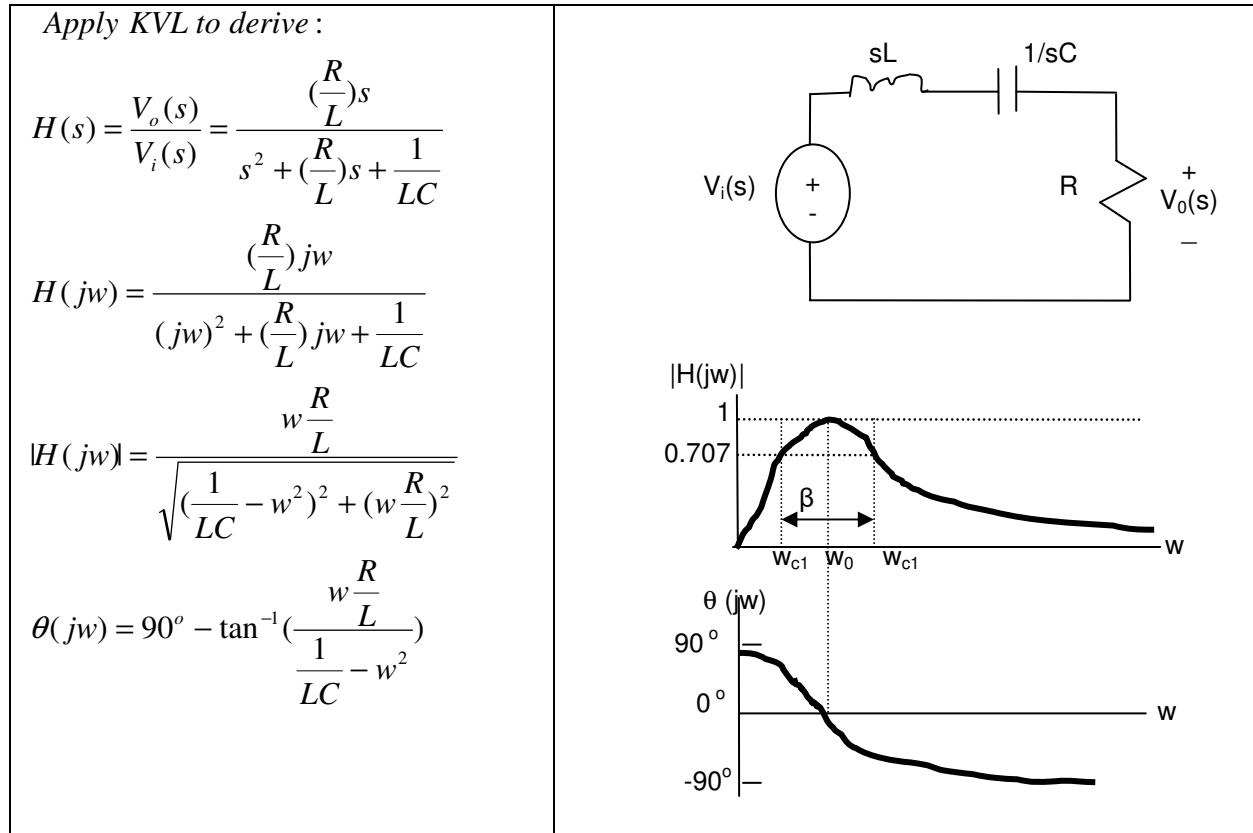
$$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}}.$$

Defined below are two other parameters which characterize the quality of filter:

- The bandwidth, $\beta = \omega_{c2} - \omega_{c1}$
Bandwidth defines a range of frequencies that signal passes through the filter.
- Quality factor, Q
Quality factor is defined as the ratio of center frequency to Bandwidth ($Q = \omega_0 / \beta$). Quality factor (Q) is a measure of filter quality since Q normalizes the bandwidth with respect to its pass band frequency.

❖ Band Pass Filter using series RLC circuit

This section analyzes the band pass filter using series RLC design and derives corresponding transfer function, cut off frequencies, Bandwidth and quality factor.



Calculate Key Band pass filter parameters :

1) ω_0 occurs when total impedance is real then $\Rightarrow j\omega_0 L + \frac{1}{j\omega_0 C} = 0 \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}}$

2) Cutoff - Freq., ω_c occurs when $|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$ and $H_{\max} = |H(j\omega_0)| = 1$

$$\frac{1}{\sqrt{2}} = \frac{\omega_c \frac{R}{L}}{\sqrt{\left(\frac{1}{LC} - \omega_c^2\right)^2 + \left(\omega_c \frac{R}{L}\right)^2}} \rightarrow \omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \quad \text{and} \quad \omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

Note that $\omega_0 = \sqrt{\omega_{c1}\omega_{c2}} = \sqrt{\frac{1}{LC}}$ same as before

3) Bandwidth $\rightarrow \beta = \omega_{c2} - \omega_{c1} = \frac{R}{L}$

4) Quality Factor $\Rightarrow Q = \frac{\omega_0}{\beta} \Rightarrow Q = \sqrt{\frac{L}{CR^2}}$

implementation Independent $H(s), w_{c1}, w_{c2}, :$

$$H(s) = \frac{\beta s}{s^2 + \beta s + w_0^2}; \quad w_{c1} = w_0 \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]; \quad w_{c2} = w_0 \left[+\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]$$

❖ Band Pass Filter using parallel RLC circuit

This section analyzes the band pass filter using parallel RLC design and derives corresponding transfer function, center frequency, cut off frequencies, Bandwidth and quality factor.

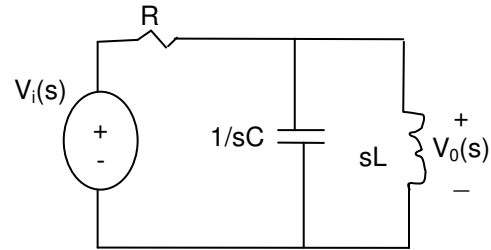
Apply KVL to derive:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s/RC}{s^2 + s/RC + 1/LC}$$

$$\omega_o = \sqrt{1/LC}$$

$$\beta = 1/RC$$

$$Q = \omega_o / \beta$$



Implementation Independent $H(s)$, ω_{c1} , ω_{c2} :

$$H(s) = \frac{\beta s}{s^2 + \beta s + \omega_o^2}; \quad \omega_{c1} = \omega_o \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]; \quad \omega_{c2} = \omega_o \left[+\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]$$

➤ Example

Design a filter that has pass band from 100 to 400 rad./sec.

Solution:

➤ Example

Design a filter that detects whistling.

Solution:

➤ Example

Design a filter that shuts off lights when there is no sound ranging from 10 Hz to 20 kHz present.

Solution:

➤ Example

Design a band pass filter with:

$$\begin{aligned}W_0 &= 200,000 \text{ rad/sec} \\B &= 500 \text{ rad/sec}\end{aligned}$$

Solution:

12.5. Band Reject Filters

Band reject, like band pass filter, is characterized by five parameters: 2 cut off frequencies, center frequency, bandwidth and quality factor. Again only 2 of the 5 parameters (W_{c1} , W_{c2} , W_0 , β and Q) can be specified independently and the rest can be derived.

❖ Band Reject Filter Using Series RLC

This section analyze the band reject filter using series RLC design and derives corresponding transfer function, center frequency, cut off frequencies, Bandwidth and quality factor.

Apply KVL to derive:

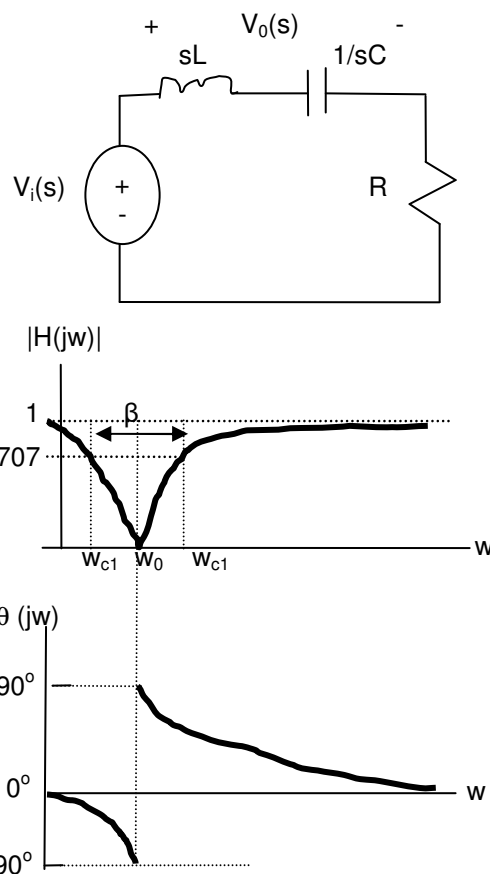
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{sL + \frac{1}{sC}}{sL + \frac{1}{sC} + R}$$

$$H(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$H(j\omega) = \frac{\frac{1}{LC} - \omega^2}{(\frac{\omega R}{L})j + (\frac{1}{LC} - \omega^2)}$$

$$|H(j\omega)| = \frac{|\frac{1}{LC} - \omega^2|}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + (\frac{\omega R}{L})^2}}$$

$$\theta(j\omega) = -\tan^{-1} \left(\frac{\frac{\omega R}{L}}{\frac{1}{LC} - \omega^2} \right)$$



Calculate Key filter parameters :

$$1) w_0 \text{ occurs when total impedance is real then } \Rightarrow jw_0 L + \frac{1}{jw_0 C} = 0 \Rightarrow w_0 = \sqrt{\frac{1}{LC}}$$

$$2) \text{ Cutoff - Freq., } w_c \text{ occurs when } |H(jw_c)| = \frac{1}{\sqrt{2}} H_{\max} \text{ and } H_{\max} = |H(jw_0)| = 1$$

$$\frac{1}{\sqrt{2}} = \frac{\left| \frac{1}{LC} - w_c^2 \right|}{\sqrt{\left(\frac{1}{LC} - w_c^2 \right)^2 + \left(w_c \frac{R}{L} \right)^2}} \Rightarrow w_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L} \right)^2 + \left(\frac{1}{LC} \right)} \quad \text{and} \quad w_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L} \right)^2 + \left(\frac{1}{LC} \right)}$$

$$3) \text{ Bandwidth } \rightarrow \beta = w_{c2} - w_{c1} = \frac{R}{L}$$

$$4) \text{ Quality Factor } \rightarrow Q = \frac{w_0}{\beta} = \sqrt{\frac{L}{R^2 C}}$$

implementation Independent $H(s)$, w_{c1} , w_{c2} :

$$H(s) = \frac{s^2 + w_0^2}{s^2 + \beta s + w_0^2}; \quad w_{c1} = w_0 \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]; \quad w_{c2} = w_0 \left[+\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]$$

❖ Band Reject Filter Using Parallel RLC

This section analyzes the band reject filter using parallel RLC design and deriving the corresponding transfer function, cut off frequencies, Bandwidth and quality factor.

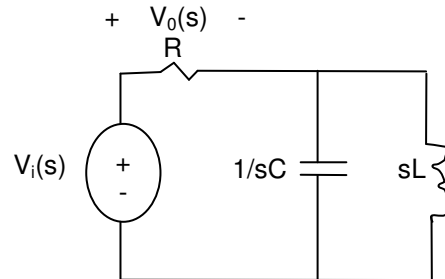
Apply KVL to derive:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s^2 + 1/LC}{s^2 + s/RC + 1/LC}$$

$$\omega_o = \sqrt{1/LC}$$

$$\beta = 1/RC$$

$$Q = \frac{\omega_o}{\beta} = \sqrt{\frac{R^2 C}{L}}$$



Implementation Independent $H(s)$, ω_{c1} , ω_{c2} :

$$H(s) = \frac{s^2 + \omega_o^2}{s^2 + \beta s + \omega_o^2}; \quad \omega_{c1} = \omega_o \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]; \quad \omega_{c2} = \omega_o \left[+\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]$$

➤ Example

Design a band reject filter with:

$$\omega_o = 200,000 \text{ rad/sec}$$

$$B = 500 \text{ rad/sec}$$

Solution:

12.6. Bode Plots

A Bode plot, named after Hendrik Wade Bode, is a logarithmic plot of either magnitude or phase of the transfer function ($H(j\omega)$) versus frequency (ω). It is intended to approximate the characteristic of a given filter circuit without the need to calculate every point and plot it. This is a useful tool for engineers to estimate filter behavior during the design phase. For an exact plot of the circuit transfer function, designer may use a modeling tool such as Mat lab.

Before starting with Bode plot, we need to do a brief review of common log definitions and operations. Log is defined by the following relationship:

$$\text{Log}_b a = x \Rightarrow a = b^x \text{ where}$$

“a” is the value being logged

“b” is the log base. In the following special cases, base is not shown:

“base 10” is written as $\log(a) = x \rightarrow a = 10^x$

“base e” is called natural log and is written as $\ln(a) = x \rightarrow a = e^x$

Below are some of the most common log operations:

$$\text{Log}(ab) = \text{Log}(a) + \text{Log}(b)$$

$$\text{Log}(a/b) = \text{Log}(a) - \text{Log}(b)$$

$$\text{Log}(a^n) = n (\text{Log}(a))$$

Here are three trivial log calculations to remember:

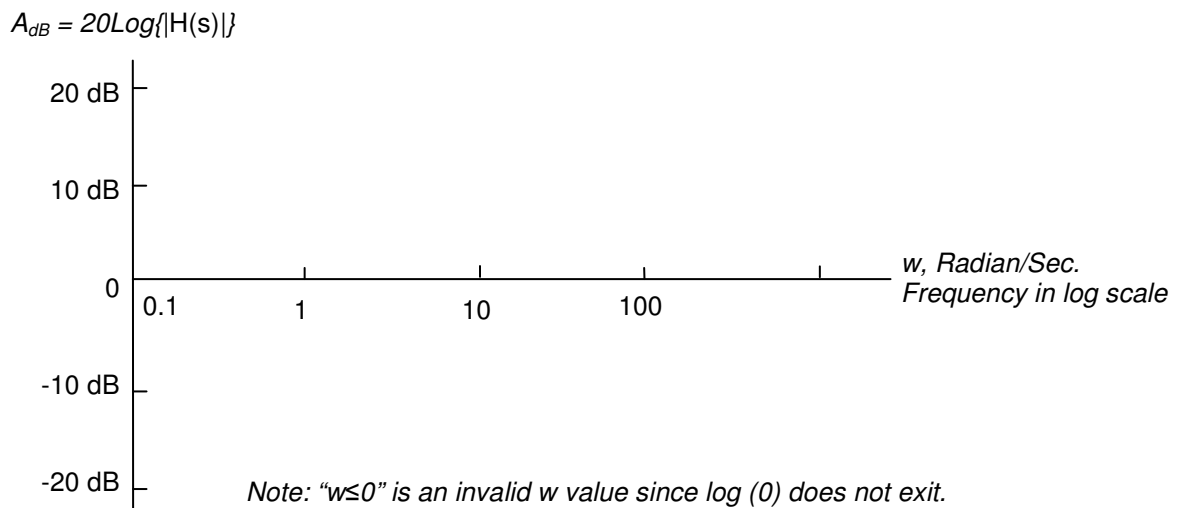
$$\text{Log}_{10} 10 = 1$$

$$\text{Log}_{10} 1 = 0$$

$$\text{Log}_{10} 0 \leq \text{Invalid}$$

The remainder of this section describes the process of drawing a magnitude Bode plots based on the circuit transfer function $\{H(s) = V_o(s)/V_i(s)\}$.

Bode plot's horizontal axis represent frequency (ω radians/second) in logarithmic scale. The vertical axis of Bode plot is the value of A_{dB} which is $\{A_{dB} = 20 \log |H(s)|\}$. Here is a typical Bode plot axis and labels:



Now, let's explore the relationship between $H(s)$ and A_{dB} on the Bode Plot. A typical transfer function may have the following form:

$$H(s) = \frac{K(s + z_1)}{s(s + p_1)} \rightarrow H(jw) = \frac{K(jw + z_1)}{jw(jw + p_1)}$$

$$\text{Step1) Convert to Standard Form } H(jw) = \frac{Kz_1(1 + jw/z_1)}{p_1(jw)(1 + jw/p_1)}$$

$$\text{Let } K_o = \frac{Kz_1}{p_1} \text{ which is a constant } \therefore$$

$$H(jw) = \frac{K_o |1 + jw/z_1| \angle \phi_1}{|w| \angle 90 |1 + jw/p_1| \angle \beta_1} = \frac{K_o |1 + jw/z_1|}{|w| |1 + jw/p_1|} \angle \phi_1 - 90 - \beta_1$$

$$\text{where: } \phi_1 = \tan^{-1} w/z_1 \quad \beta_1 = \tan^{-1} w/p_1$$

Next, evaluate the value $A_{dB} = 20 \log |H(s)|$ where units are decibels (dB)

$$A_{dB} = 20 \log_{10} |H(jw)|$$

$$A_{dB} = 20 \log_{10} \frac{K_o |1 + jw/z_1|}{|w| |1 + jw/p_1|} \quad \text{where } K_o = \frac{Kz_1}{p_1} \text{ which is a constant } \therefore$$

$$A_{dB} = 20 \log_{10} K_o + 20 \log_{10} |1 + jw/z_1| - 20 \log_{10} w - 20 \log_{10} |1 + jw/p_1|$$

Look at each term and then combine the graphs from each term

$$20 \log_{10} K_o \Rightarrow \text{straight line : (+Value if } K_o > 0) \text{ (equals 0 if } K_o = 1) \text{ (-Value if } K_o < 1)}$$

$$20 \log_{10} |1 + jw/z_1| \rightarrow 0 \text{ as } w \rightarrow 0$$

$$20 \log_{10} |1 + jw/z_1| \rightarrow 20 \log_{10} (w/z_1) \text{ as } w \rightarrow \infty$$

General form of $H(s)$ is shown below:

$$H(s) = \frac{K_o(s + z_1)(s + z_2)(s + z_3) \dots}{s(s + p_1)(s + p_2)(s + p_3) \dots}$$

Leveraging the log operations ($\log a/b = \log a - \log b$) as done in the earlier example, we can turn all these multiplications and divisions to additions and subtractions as shown below:

$$A_{dB} = 20 \log_{10} K_o + 20 \log_{10} |1 + jw/z_1| + \dots - 20 \log_{10} w - 20 \log_{10} |1 + jw/p_1| - \dots$$

Each term adds to or subtracts from the slop of Bode plot as shown below:

- $K_o \Rightarrow 20 \log_{10} K_o$
This is a horizontal line on Bode plot representing constant value.
- $1/s \Rightarrow 20 \log_{10} w$
This term is a straight line with a slope of -20 dB/decade that intersects the 0 dB axis at $w=1$.
Note: A decade is 1-to-10 change in frequency. For example $w=100$ to $w=1000$ is a decade.
- $(s + z_1) \Rightarrow 20 \log_{10} |1 + jw/z_1|$
This term is a straight line with a slope of 20 dB/decade that intersects the 0 dB axis at $w=z_1$.
- $1/(s + p_1) \Rightarrow -20 \log_{10} |1 + jw/p_1|$
This term is a straight line with a slope of -20 dB/decade that intersects the 0 dB axis at $w=p_1$.

❖ Example – Bode plot

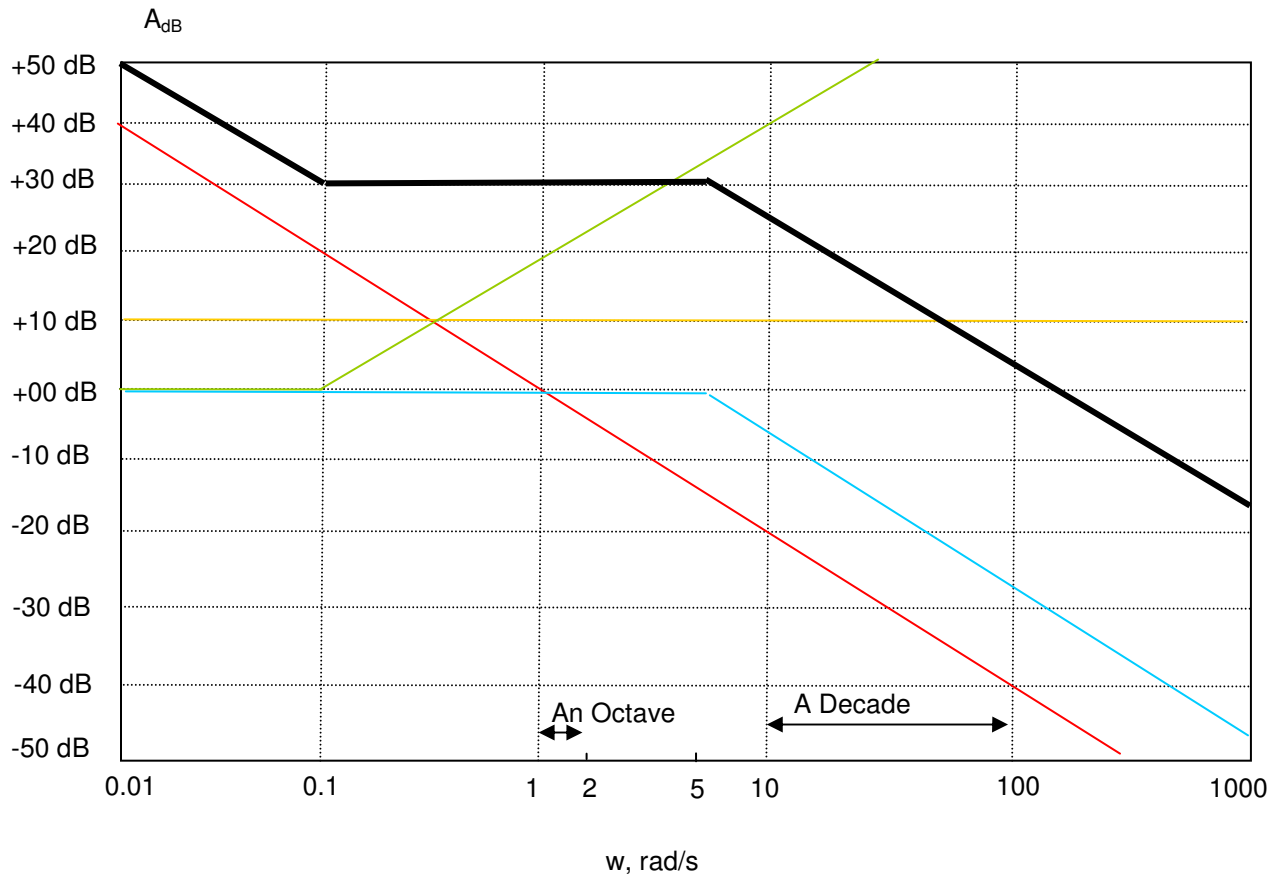
Graph the Bode plot for $H(s) = \frac{\sqrt{10}(s+0.1)}{s(s+5)}$ when:

$$k_0 = \sqrt{10} \rightarrow A_{dB} = 20 \log_{10}(10)^{0.5} \quad \text{Note } k_0 = \frac{Kz_1}{p_1}$$

$z_1 = 0.1 \text{ rad / Sec.} \rightarrow 20\text{dB per Decade with } 0 \text{ dB axis intersect at } w = z_1 (0\text{dB when } w < z_1)$

$-20\text{Log}_{10}w \rightarrow -20\text{dB per Decade with } 0 \text{ dB axis intersect at } w = 1$

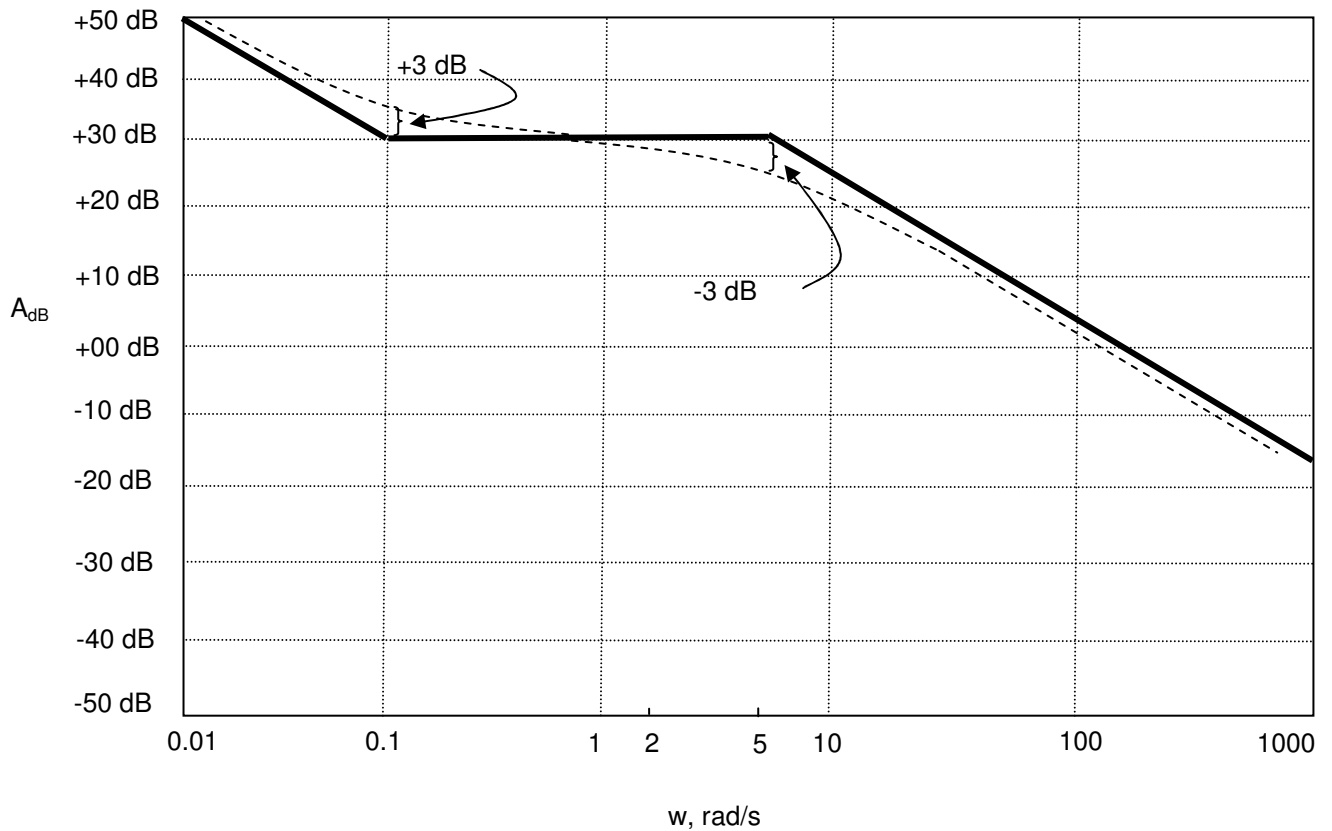
$p_1 = 5 \text{ rad / Sec.} \rightarrow -20\text{dB per Decade with } 0 \text{ dB axis intersect at } w = p_1 (0\text{dB when } w < p_1)$



- Example
if the input to the circuit represented by the above Bode plot is $v(t)=10\sin(10wt)$ volts, what is the output at $w=100 \text{ rad/sec}$ in volts.

Solution:

A better approximation may be achieved by a 3 dB at the cut-off frequencies as shown below:



As discussed earlier $3dB \approx 10 \log \frac{1}{\sqrt{2}}$ improves the graph's approximation of the real filter transfer function.

This is due to the fact the 3dB represent the point where power is one half of maximum power at cut-off frequency.

❖ Example – Bode plot

Graph the Bode plot for $H(s) = \frac{10^9}{(1000 + s)^3}$

Solution:

"Student Exercise"

❖ Example – Bode plot

Graph the Bode plot for $H(s) = \frac{s(s+100)(s+1,000)}{(s+5,000)(s+10,000)}$

Solution:

“Student Exercise”

❖ Example – Bode plot

For the transfer function, $H(s) = \frac{10^3(s+100)}{s(s+10)(s+1000)}$

a) draw Bode Plot.

b) find output voltage magnitude if the input in $v_i(t) = 25 \cos(100t - \pi/2)$.

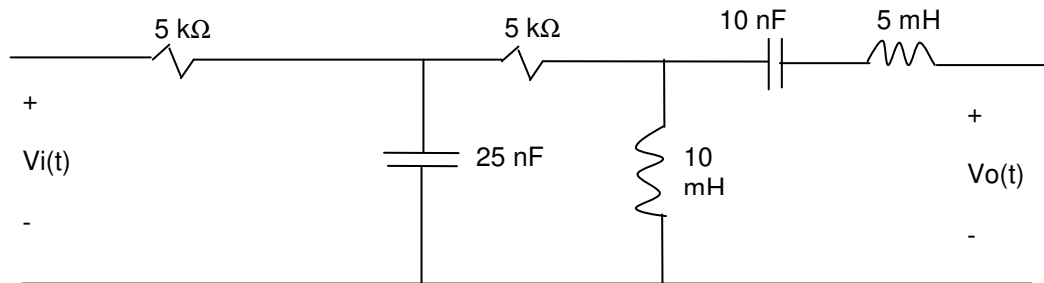
c) find output voltage magnitude if the input in $v_i(t) = 25 \sin(10,000t + \pi/3)$.

Solution:

“Student Exercise”

❖ Example – Bode plot

For the following filter circuit use a bode plot to graph and describe its response (functionality):

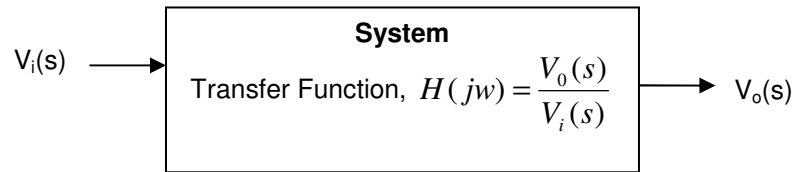


Solution:

"Student Exercise"

12.7. Summary

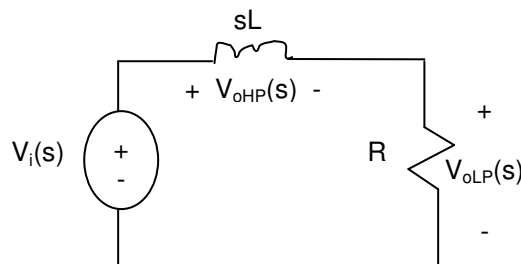
❖ Transfer Function and Cut-off Frequency



Cut-off frequency, ω_c , satisfies the following equation:

$$|H(j\omega_c)| = \frac{H_{\max}}{\sqrt{2}}$$

❖ RL Circuit, $\left[\omega_c = \frac{R}{L} \right]$



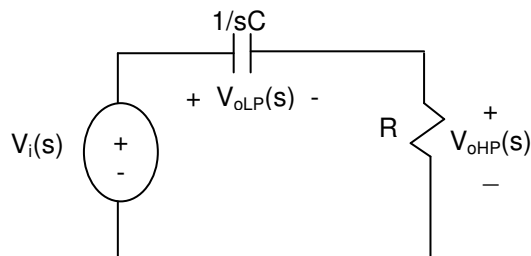
Low Pass Filter:

$$H_{LP}(s) = \frac{V_{oLP}(s)}{V_i(s)} = \frac{\omega_c}{s + \omega_c}$$

High Pass Filter:

$$H_{HP}(s) = \frac{V_{oHP}(s)}{V_i(s)} = \frac{s}{s + \omega_c}$$

❖ RC Circuit, $\left[\omega_c = \frac{1}{RC} \right]$



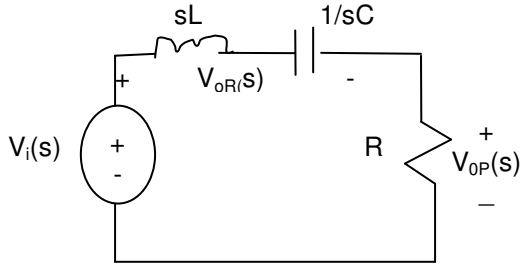
Low Pass Filter:

$$H_{LP}(s) = \frac{V_{oLP}(s)}{V_i(s)} = \frac{\omega_c}{s + \omega_c}$$

High Pass Filter:

$$H_{HP}(s) = \frac{V_{oHP}(s)}{V_i(s)} = \frac{s}{s + \omega_c}$$

- ❖ RLC Series Circuit, $\left[w_o = \frac{1}{\sqrt{LC}} \quad \& \quad \beta = \frac{R}{L} \right]$



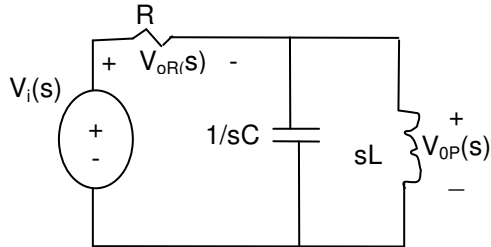
Band Pass Filter:

$$H_{LP}(s) = \frac{V_{OP}(s)}{V_i(s)} = \frac{\beta s}{s^2 + \beta s + w_o^2}$$

Band Reject Filter:

$$H_{LP}(s) = \frac{V_{OR}(s)}{V_i(s)} = \frac{s^2 + w_o^2}{s^2 + \beta s + w_o^2}$$

- ❖ RLC Parallel Circuit, $\left[w_o = \frac{1}{\sqrt{LC}} \quad \& \quad \beta = \frac{1}{RC} \right]$



Band Pass Filter:

$$H_B(s) = \frac{V_{OP}(s)}{V_i(s)} = \frac{\beta s}{s^2 + \beta s + w_o^2}$$

Band Reject Filter:

$$H_R(s) = \frac{V_{OR}(s)}{V_i(s)} = \frac{s^2 + w_o^2}{s^2 + \beta s + w_o^2}$$

- ❖ Quality Factor

$$Q = \frac{w_o}{\beta}$$

- ❖ Filter equations

Filter Type		W_0	β	Q	Wc	$H(s)$
Low Pass	RL				R/L	$\frac{w_c}{s + w_c}$
	RC				1/RC	
High Pass	RL				R/L	$\frac{s}{s + w_c}$
	RC				1/RC	
Band Pass	Series	$\sqrt{1/LC}$	R/L	W_0/β	$w_0 \left[\pm \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]$	$\frac{\beta s}{s^2 + \beta s + w_o^2}$
	Parallel		1/RC			
Band Reject	Series		R/L			$\frac{s^2 + w_o^2}{s^2 + \beta s + w_o^2}$
	Parallel		1/RC			

- ❖ Bode Plot

Graphical representation of transfer function, $H(S)$, using logarithmic frequency scale.

12.8. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 14.

12.9. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 13. Active Filters

Key Concepts and Overview

- ❖ Active vs. Passive Filter
- ❖ First-order Active Low Pass Filters
- ❖ First-order Active High Pass Filters
- ❖ Scaling Filter Elements and Parameters
- ❖ Active Band Pass Filters
- ❖ Active Band Reject Filters
- ❖ Higher Order Active Filter
- ❖ Specialized Filter Designs
- ❖ Additional Resources

13.1. Active vs. Passive Filter

This chapter covers design and analysis of active filters. Many of the concepts introduced in the previous chapter apply directly to active filters. All four basic filter types (low pass, high pass, band pass and band reject) can be designed using Op Amps, resistors and capacitors. The use of Op Amp, an active component, is the reason for naming this type of filter active.

Active filters have three distinct advantages over their equivalent passive filter designs:

- 1) All active filters can be designed without inductors. This is an advantage since inductors are large, heavy, highly variable and costly to produce. Additionally, Inductors produce an undesirable electromagnetic field effect in the circuit.
- 2) Active filters can be designed with gains larger than one which is the maximum gain possible with passive filter. In other words, active filters allow signal amplification.
- 3) Active filters' cut off frequency and magnitude of transfer function do not change with the addition of load impedance, unlike a passive filter.

In general, an active filter is preferred, due to the advantages listed above, as long as the power required for the Op Amp DC Bias is available.

The remainder of this chapter is dedicated to the analysis and design of Active filters for each of the major filter types (low pass, high pass, band pass and band reject). Unless specified otherwise, we will be using ideal Op Amp model which means Op Amp input currents $I_p = I_n = 0$, input voltages $V_p = V_n$ and open loop gain $A = \infty$.

13.2. First-Order Active Low Pass Filters

This section introduces a first-order active low pass filter design with a Bode plot of its transfer function and definition of its prototype design.

$$H(s) = \frac{V_o}{V_i} = ?$$

At V_n write KCL Equation

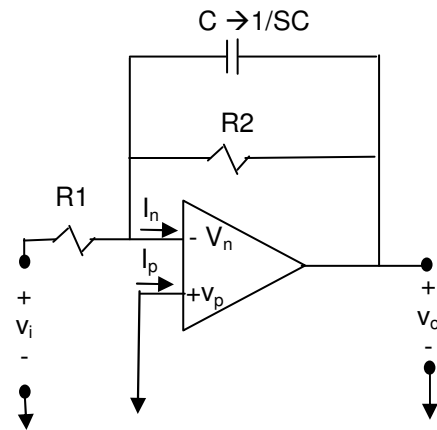
$$\frac{V_n - V_i}{R_1} + \frac{V_n - V_o}{R_2} + \frac{V_n - V_o}{1/sC} = 0$$

$$V_p = V_n = 0$$

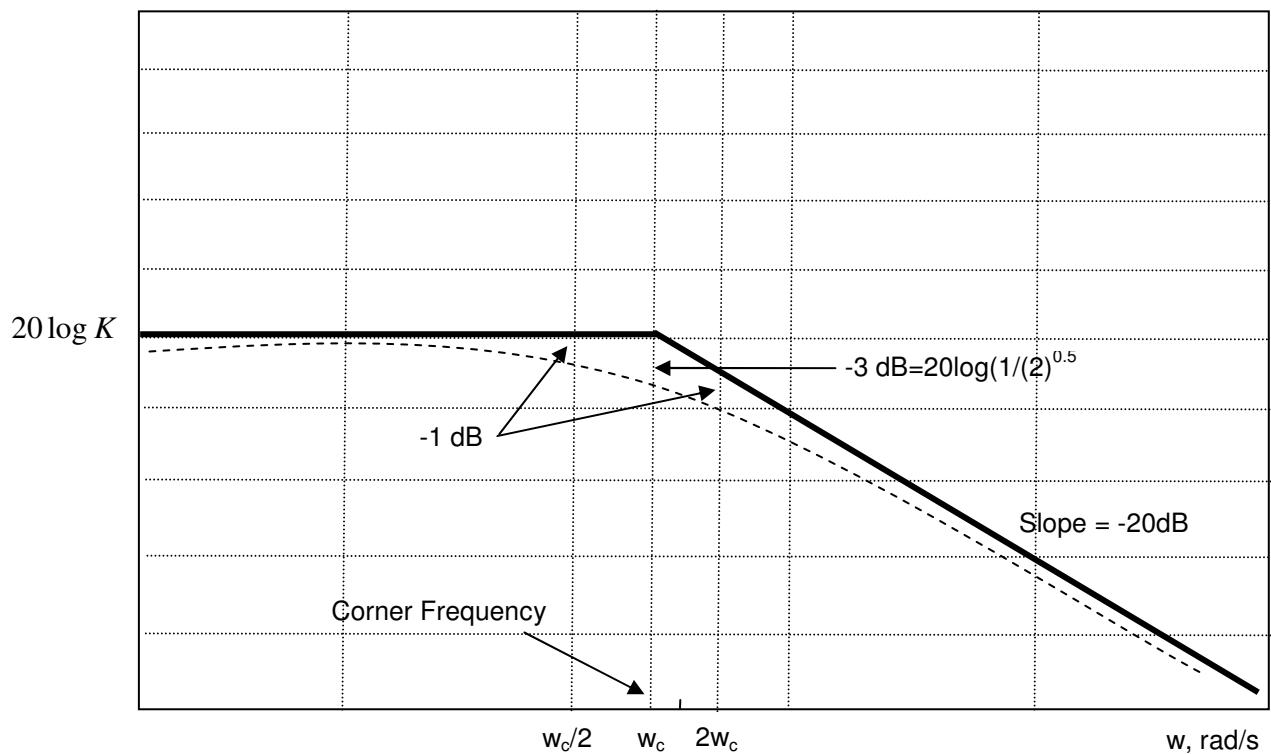
$$H(s) = \frac{V_o}{V_i} = -\frac{R_2}{R_1} \frac{1}{R_2Cs + 1}$$

$$H(s) = -K \frac{\omega_c}{s + \omega_c} \text{ where } K = \frac{R_2}{R_1} \quad \omega_c = \frac{1}{R_2C}$$

Note: Gain and Cut off frequency can be set independently; and the gain may be >1 .



$$A_{dB} = 20 \log |H(s)|$$



❖ Prototype low pass Op Amp filter

In general prototype filter is a filter where the gain $K=1$ and cut off frequency $\omega_c=1$. Therefore, a

prototype low pass filter may be defined by:

$$H(s) = (-1)/(s+1)$$

$R_1=R_2=1$ ohm and $C=1$ F.

❖ Example

Design an active low pass filter with a gain of 230 and cut off frequency of 2000 Hz.

Solution:

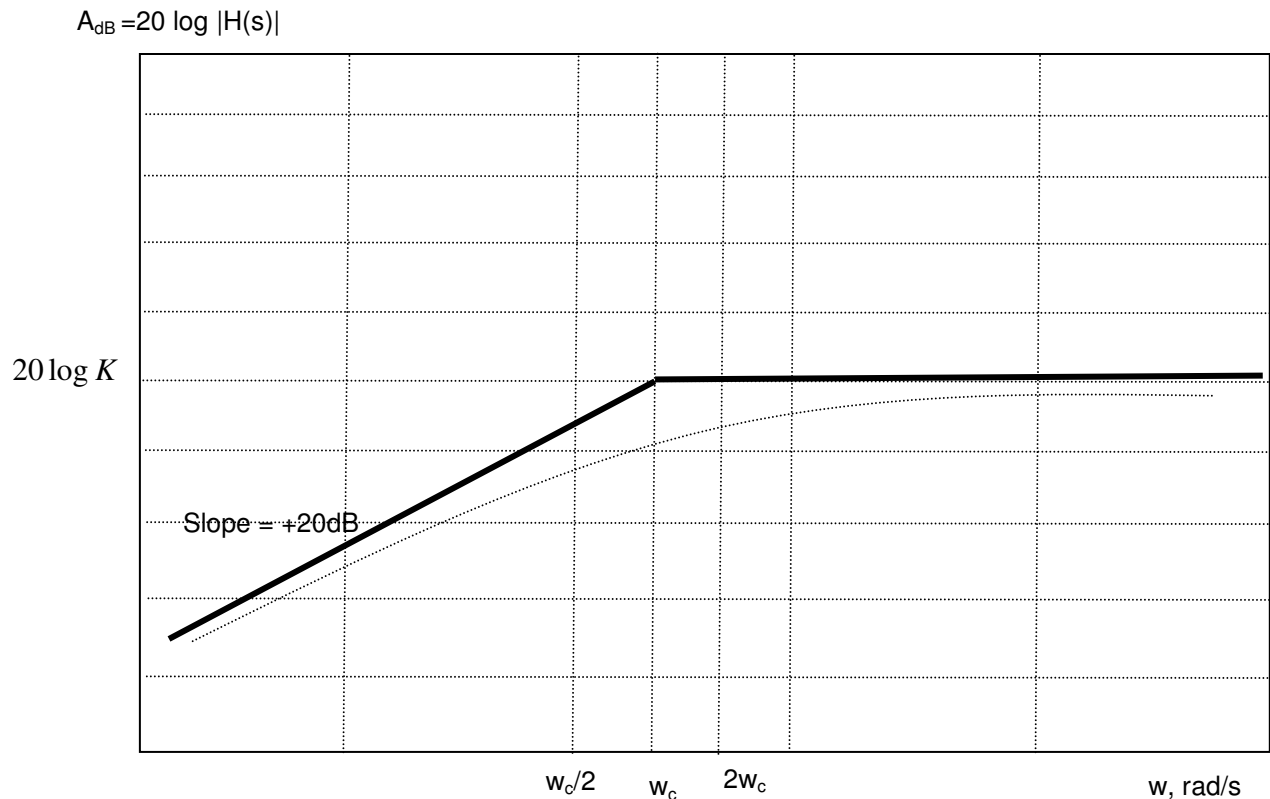
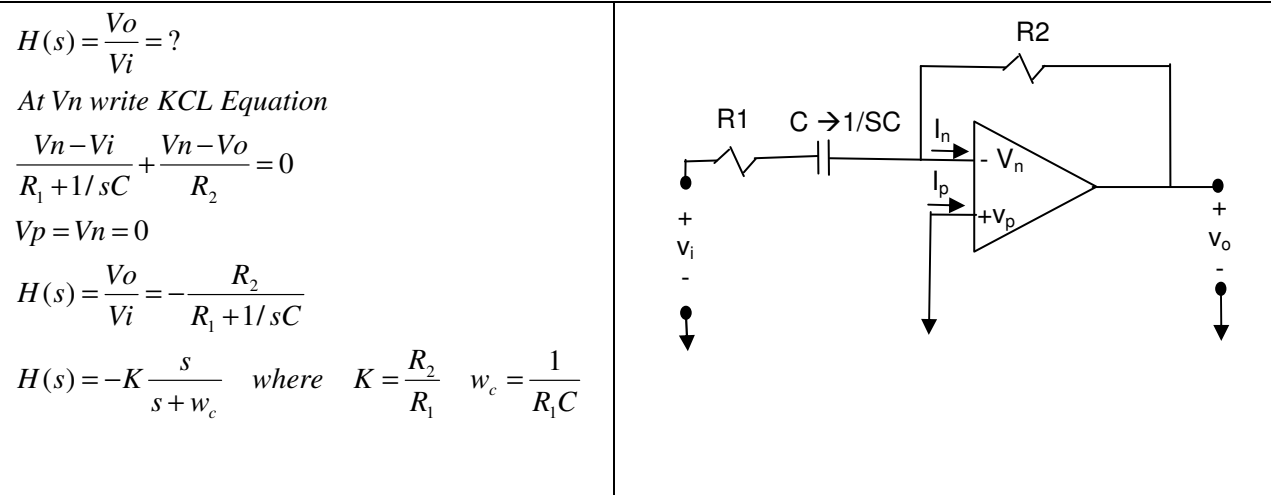
❖ Example

Design an active filter with transfer function: $H(s) = -\frac{10^6}{s + 5000}$.

Solution:

13.3. First-Order Active High Pass Filters

This section introduces a first-order active high pass filter design, Bode plot of its transfer function and definition of its prototype design.



- ❖ **Prototype High Pass Op Amp Filter**
As discussed earlier, prototype filter has a gain $K=1$ and cut off frequency $\omega_c=1$. A prototype high pass filter may be defined by:

$$H(s) = (-s)/(s+1)$$

$R_1=R_2=1 \text{ ohm}$ and $C=1 \text{ F}$.

❖ Example – Active High Pass Filter Design

Compute the values of R_2 and C that yields a high-pass filter with a pass band gain of 1 if R_1 is 1 Ω .

Solution:

$$\text{Gain}=K=R_2/R_1 = 1 \rightarrow R_2 = 1 \Omega$$

$$\text{Cut Off Frequency} = \omega_c = 1/(R_1 C) \rightarrow C = 1 \text{ F}$$

Note: This is the prototype of high pass filter (unity) where $C=1\text{F}$, $R_1 = R_2 = 1 \Omega$

All the other filter types such as band pass and band reject may be designed using a combination of low and high pass filter. Next section introduces scaling process before discussing other types of filters.

❖ Example

Design an active low pass filter using 10,000 Ω resistors with a cut off of 50 krad/s. Start with a prototype filter.

Solution:

13.4. Scaling Filter Elements and Parameters

This section introduces the concept of filter design scaling with respect to cut off frequency and magnitude of transfer function. The scaling process applies to both active and passive filters. By the definition component scaling factor K_g and frequency scaling factor K_f are positive. These parameters scale the value of capacitors, inductors, resistors and frequencies.

In the following analysis the prime is used to identify scaled parameters (R' , L' , C' , w') and the original parameters are shown as not primed (R , L , C , w). Active and passive design differ only in the fact that passive filter gain is less than 1 (due to loss) and active filter gain is equal to the ratio of feedback impedance and source impedance control the gain.

The following steps outline the process of scaling factor application:

- Select the cut off frequency, w_c for low/high pass or center frequency, w_o for band pass/band reject filters at 1 rad/s
- Select gain, $K = 1$
- Select $C = 1$ F
- Calculate the value of resistors needed to reach the desired pass band gain at 1 rad/s cut-off or center frequency.
- Use the scaling steps below to compute component values that provide the desired cut off or center frequency & gain. *Note that original component values are without prime (R , L , C , w_o) and scaled component values are shown with prime (R' , L' , C' , w').*
 - Component Value Scaling
 K_c is the component value scaling factor which allows changes in value of component without impacting the frequency or the filter transfer function.

$$R' = K_c R, \quad L' = K_c L, \quad C' = C/K_c \rightarrow K_c = R'/R = L'/L = C/C'$$
 - Frequency Scaling
 K_f is frequency scaling factor ($K_f = w_c'/w_c$) for low or high pass and ($K_f = w_o'/w_o$) for band reject or band pass filters. By adjusting based on as shown below, the cut off frequency is adjusted to the desired value.

$$R' = R, \quad L' = L/K_f, \quad C' = C/K_f \rightarrow K_f = L/L' = C'/C$$
 - Simultaneous Frequency and component value scaling

$$R' = K_c R, \quad L' = (K_c/K_f)L, \quad C' = C/(K_c K_f)$$

❖ Example – Scaling

What scale factors will transform the prototype high-pass filter into a high-pass filter with a 0.5 μ F capacitor and a cutoff frequency of 10 KHz.

Solutions:

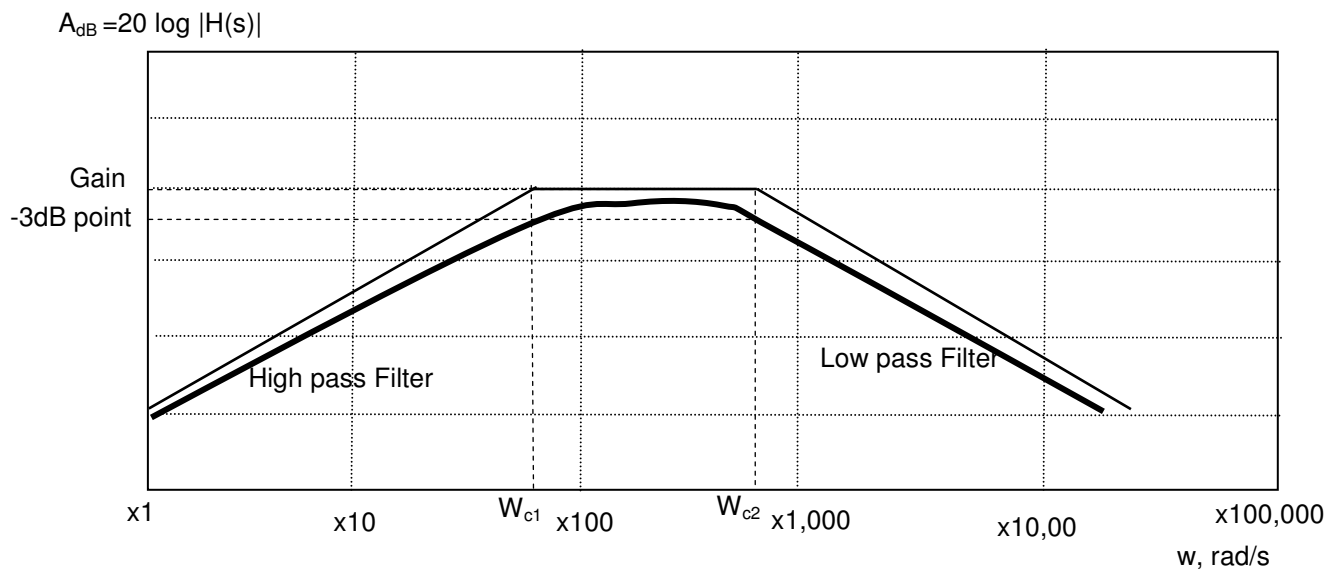
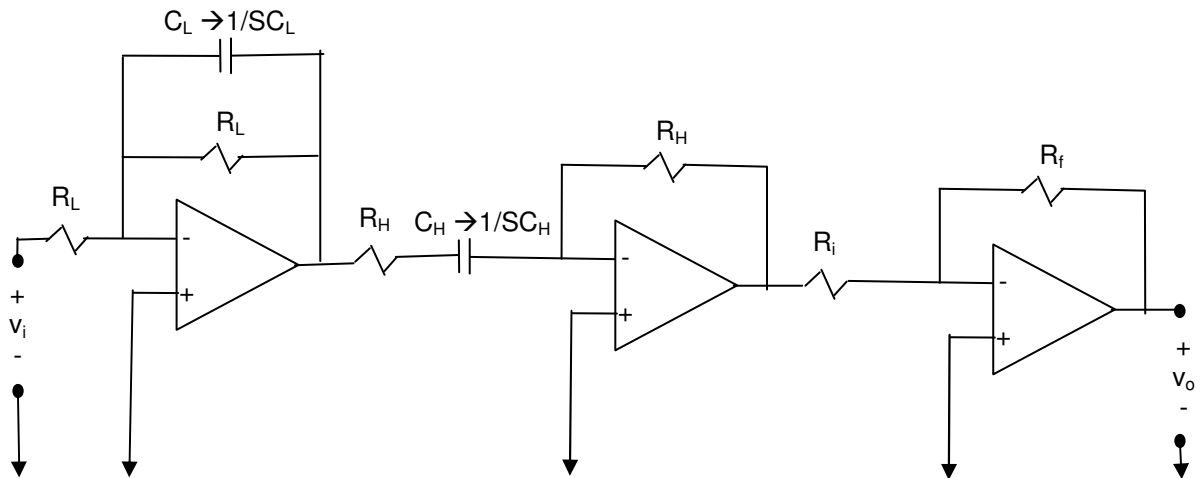
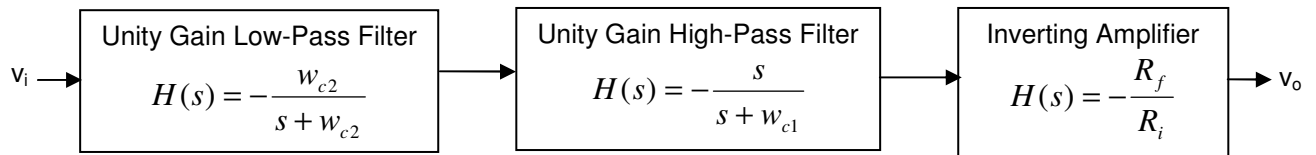
We know for the prototype high-pass filter $w_c = 1$ rad/sec and $C = 1$ F

Frequency Scale Factor $\rightarrow K_f = w_o'/w_o = 20,000 \pi / 1 = 62,831.85$

Component Value Scale Factor $\rightarrow K_c = C/(K_f C') = 1 / (62,831.85 \times 0.5 \times 10^{-6}) = 31.831$

13.5. Active Band Pass Filter (Multi-stage Cascading Filter)

As discussed earlier active band pass filter can be designed using multiple low pass and high pass filters. The gain may be controlled using an amplifier stage. Below is example of a prototype (Unity) band pass filter block diagram, circuit and Bode plot of the magnitude of transfer function:



The transfer function of the above active band pass filter may be calculated by multiply transfer function of each of the stages as shown below:

$$H(s) = \frac{Vo}{Vi} = \left(-\frac{w_{c2}}{s + w_{c2}} \right) \left(-\frac{s}{s + w_{c1}} \right) \left(-\frac{R_f}{R_i} \right) = -\frac{Kw_{c2}s}{(s + w_{c1})(s + w_{c2})}$$

The Bode plot and transfer function show that the cut-off frequency, W_{c1} , is equal to the high pass filter stage cut off frequency while the cut-off frequency, W_{c2} , is equal to the low pass filter cut off frequency.

The above process of treating each stage independently is only valid if the W_{c2} is much larger than W_{c1} ($W_{c2} \gg W_{c1}$) which means that W_{c2} is at least twice the value of W_{c1} ($W_{c2} \geq W_{c1}$). Using the assumption that $W_{c2} \gg W_{c1}$ the transfer function for $H(s)$ may be rewritten as:

$$H(s) = -\frac{Kw_{c2}s}{(s + w_{c1})(s + w_{c2})}$$

Where :

$$w_{c2} = w_{cL} = \frac{1}{R_L C_L} \quad \text{Low pass filter cut-off frequency}$$

$$w_{c1} = w_{cH} = \frac{1}{R_H C_H} \quad \text{High pass filter cut-off frequency}$$

$$w_o = \sqrt{w_{c1}w_{c2}} \quad \text{Center frequency by definition}$$

$$K = \frac{R_f}{R_i} \quad \text{define the gain}$$

❖ Example – Active Band pass Filter Design

Design a filter that can pass signals between 250,000 to 500,000 Hz with a gain of 3. The Circuit can only use 10 pF capacitors.

Solution:

Given: $f_{c1} = 250$ kHz, $f_{c2} = 500$ kHz, $K=3$, $C=10$ pF and all resistors.

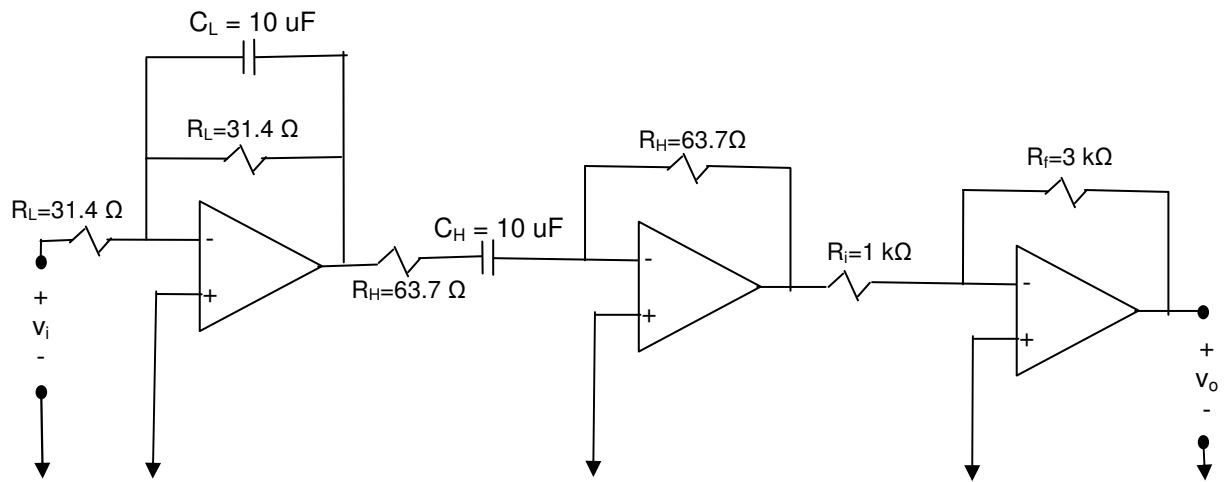
Find: Filter Design

$$w_{c2} = w_{cL} = \frac{1}{R_L C_L} \Rightarrow 2\pi \cdot 500,000 = \frac{1}{R_L \cdot 10 \times 10^{-9}} \Rightarrow R_L = 31.4 \, \Omega$$

$$w_{c1} = w_{cH} = \frac{1}{R_H C_H} \Rightarrow 2\pi \cdot 250,000 = \frac{1}{R_H \cdot 10 \times 10^{-9}} \Rightarrow R_H = 63.7 \, \Omega$$

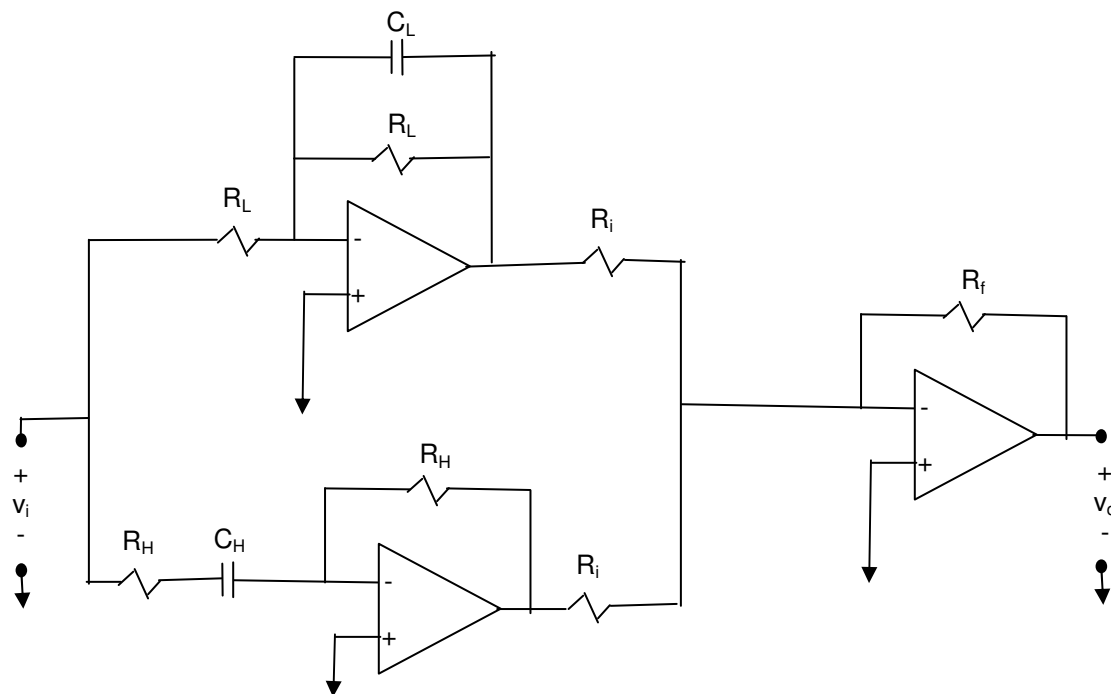
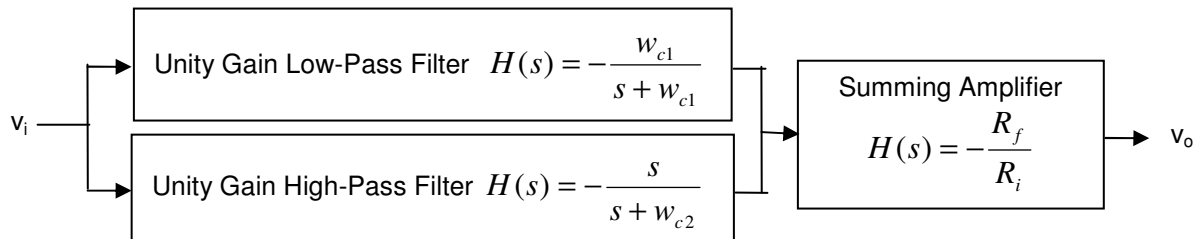
$$K = \frac{R_f}{R_i} \Rightarrow 3 = \frac{R_f}{R_i}$$

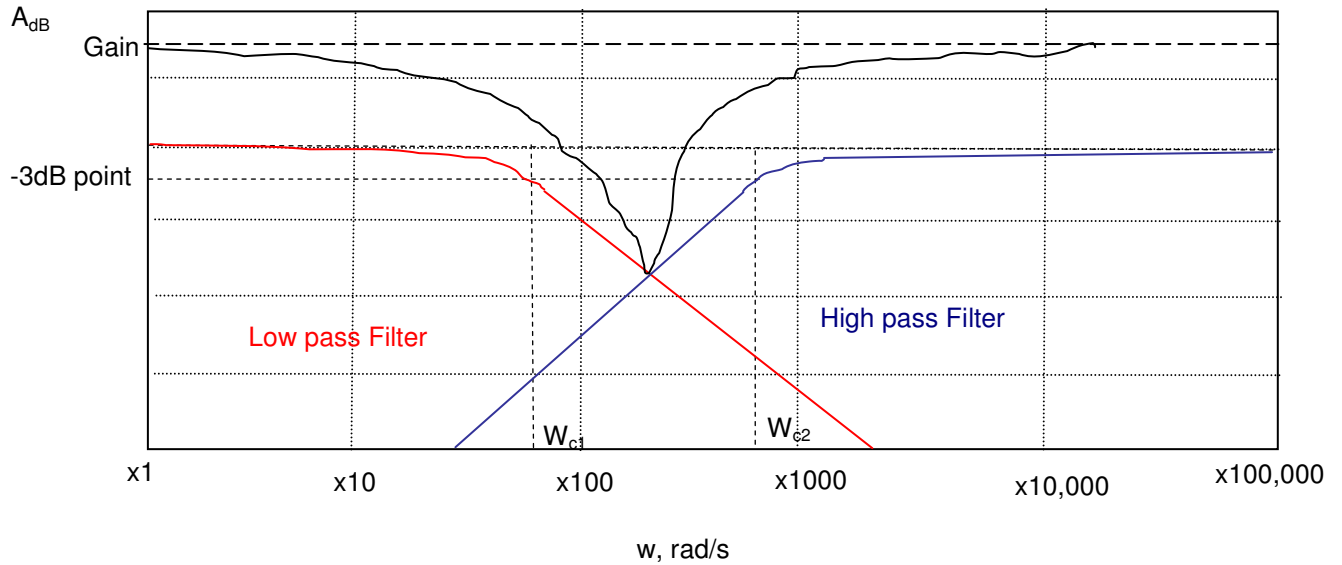
$$\text{Assume } R_i = 1 \, k\Omega \Rightarrow R_f = 3 \, k\Omega$$



13.6. Active Band Reject Filter (Multi-stage cascading filter)

As discussed earlier active band reject filter can be designed using multiple low pass and high pass filters. The gain may be controlled using an amplifier stage. Below is an example of a prototype (Unity) band pass filter block diagram, circuit, and Bode plot of the magnitude of transfer function:





The bode plot and transfer function shows that the cut-off frequency, W_{c1} , is equal to the low pass filter stage cut off frequency, while the cut-off frequency, W_{c2} , is equal to the high pass filter cut off frequency.

$$H(s) = \frac{V_o}{V_i} = \left(\frac{-w_{c1}}{s + w_{c1}} + \frac{-s}{s + w_{c2}} \right) \left(-\frac{R_f}{R_i} \right) = \left(\frac{R_f}{R_i} \right) \left(\frac{s^2 + 2w_{c1}s + w_{c1}w_{c2}}{(s + w_{c1})(s + w_{c2})} \right)$$

The above process of treating each stage independently is only valid if the W_{c2} is much larger than W_{c1} ($W_{c2} \gg W_{c1}$) which means that W_{c2} is at least twice the value of W_{c1} ($W_{c2} \geq 2W_{c1}$). Using the assumption that $W_{c2} \gg W_{c1}$ the transfer function for $H(s)$ may be rewritten as:

$$H(s) = K \frac{s^2 + 2w_{c1}s + w_{c1}w_{c2}}{(s + w_{c1})(s + w_{c2})}$$

Where :

$$w_{c1} = w_{cL} = \frac{1}{R_L C_L} \quad \text{Low pass filter cut-off frequency}$$

$$w_{c2} = w_{cH} = \frac{1}{R_H C_H} \quad \text{High pass filter cut-off frequency}$$

$$w_0 = \sqrt{w_{c1}w_{c2}} \quad \text{Center frequency by definition}$$

$$K = \frac{R_f}{R_i} \quad \text{Defines the gain}$$

❖ Example – Active Band Reject Filter Design

Design a filter that only passes through signals with frequencies below 10^6 rad/sec and signals with frequencies above 10^8 rad/sec. The only resistors available are 10 kΩ resistors but you can use various capacitors in the design.

Solution:

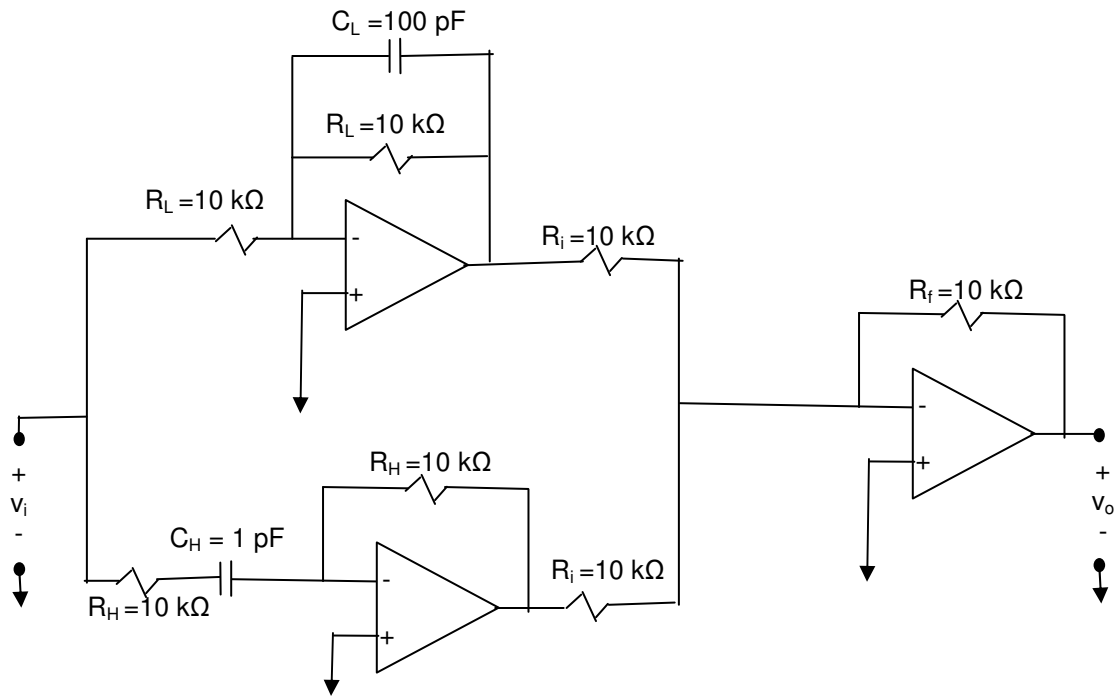
Given: $\omega_{c1} = 10^6$ rad/sec, $\omega_{c2} = 10^8$ rad/sec, $R = 10$ k Ω
Find: Band reject filter design.

$$\omega_{c1} = \frac{1}{R_L C_L} \Rightarrow 10^6 = \frac{1}{10^4 C_L} \Rightarrow C_L = 100 \text{ pF}$$

$$\omega_{c2} = \frac{1}{R_H C_H} \Rightarrow 10^8 = \frac{1}{10^4 C_H} \Rightarrow C_H = 1 \text{ pF}$$

$$\omega_0 = \sqrt{\omega_{c1} \omega_{c2}} \quad \text{Center frequency by definition}$$

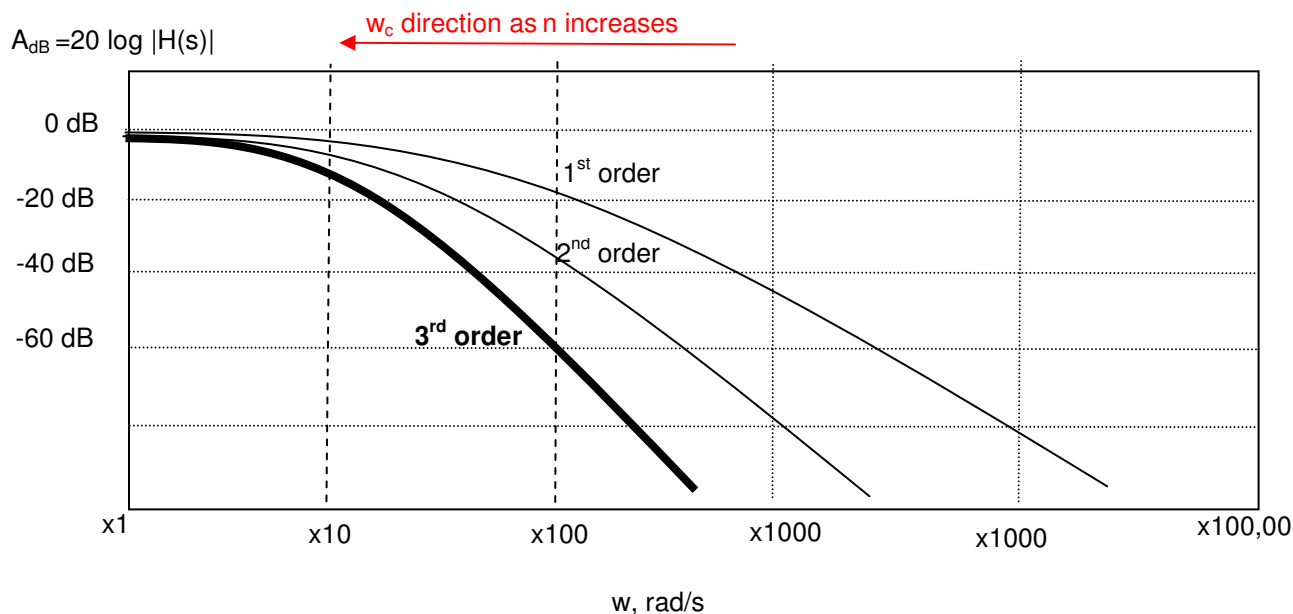
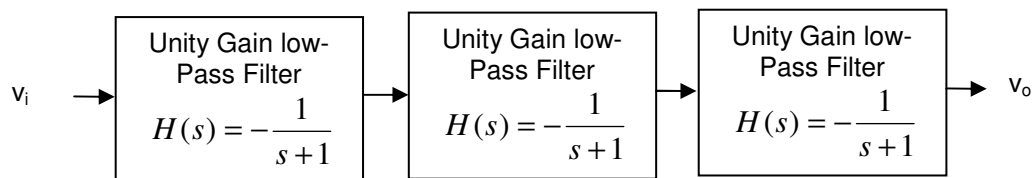
$$K = \frac{R_f}{R_i} \Rightarrow 1 = \frac{R_f}{R_i} \Rightarrow R_f = R_i = 10 \text{ k}\Omega$$



13.7. Higher Order Active filter

So far we have designed filters with transfer function delivering a “-20 dB/decade” slope transition between pass band and stop band of filters which we have referred to as 1st order. For the majority of applications a much faster transition is expected between the pass and stop band. One way to meet this need is to cascade multiple 1st order filters to increase the transition period slop. In general, an n-stage cascade filter with n identical low-pass filters will transition from pass band to stop band with a slope equal to “(-20)(n) dB/decade”

For example a three stage Low Pass filter:



Mathematics and the above Bode plot for the nth order low pass demonstrate the effects of using multiple stages of the same type filters is to create steeper transition. The only issue is that the cut-off frequency moves back closer to $w=0$ every time n increases as shown below:

For an n stage low pass filter, transfer function is derived below:

$$H(s) = \frac{V_o}{V_i} = \left(\frac{-1}{s+1} \right) \left(\frac{-1}{s+1} \right) \dots \left(\frac{-1}{s+1} \right) = \left(\frac{-1}{s+1} \right)^n = \frac{(-1)^n}{(s+1)^n}$$

Calculating the new cut – off frequency for nth order filter

$$|H(s)| = \frac{1}{\sqrt{2}}$$

$$|H(jw_{cn})| = \frac{1}{(jw_{cn} + 1)^n} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{(\sqrt{w_{cn}^2 + 1})^n} = \frac{1}{\sqrt{2}}$$

$$w_{cn} = \sqrt{(2)^{\frac{1}{n}} - 1} \text{ new cut – off frequency (3 dB point) for nth order filter}$$

The cut-off frequency of a third order prototype unity low pass filter is:

$$w_{c3} = \sqrt[3]{\sqrt{2} - 1} = 0.51 \text{ rad / sec}$$

This assumes that each stage has a cut off frequency of 1 rad/sec. If the cut off frequency of each stage is w_{c1} , the n order cut off frequency would be:

$$w_{cn} = \left[\sqrt[n]{\sqrt{2} - 1} \right] w_{c1}$$

❖ Example

Design a 4th order low pass filter with cut off frequency of 12400 rad/sec.

Solution:

❖ Example

Design a low pass filter with cut off frequency of 20,000 Hz and drop off in gain of 220 dB/decade using active low pass filters.

- a) How many stages are needed?
- b) find ω_c , R and C for one low pass filter stage.

Solution:

13.8. Specialized Filter Designs

In this chapter, we have presented generic filters for each of the four categories (low pass, high pass, band pass and band reject). There are numerous specialized filter designs, in use with each providing a unique advantage in a given application. Here are some examples with brief description:

- ❖ **Butterworth Filter Design**
The Butterworth filter design provides maximum pass band flatness which means it is optimal for applications requiring uniform gain across a range of frequencies in the pass band.
- ❖ **Tschebyscheff Filter Design**
Tschebyscheff filter design provides higher gain roll off but has more ripples in the pass band.
- ❖ **Notch Filter Design**
A notch filter is a band reject filter with a high Q factor (narrow stop band). It is commonly used in audio applications to reduce effect of feedback.
- ❖ **Bessel Filter Design**
Bessel filter design provides linear phase response (near constant group delay), which means the signal shape is preserved in the pass band.

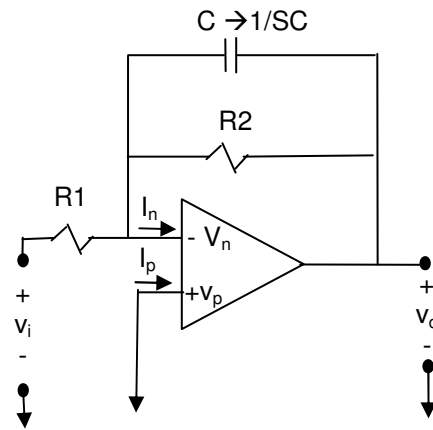
As mentioned earlier, there are numerous specialized filter designs based on the application requirements. It is important to research available designs when selecting filter design for your application.

13.9. Summary

❖ Active First Order Low Pass Filter

$$H(s) = \frac{V_o}{V_i} = -K \frac{\omega_c}{s + \omega_c}$$

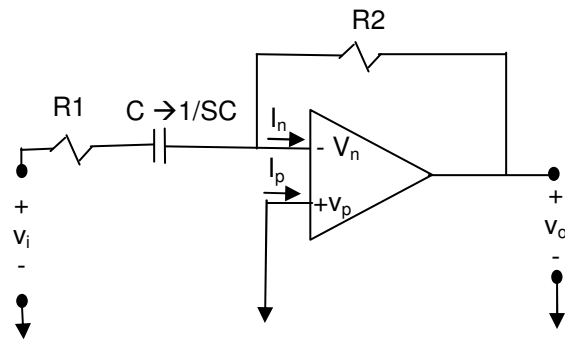
$$\text{where } K = \frac{R_2}{R_1} \quad \omega_c = \frac{1}{R_2 C}$$



❖ Active First Order High Pass Filter

$$H(s) = \frac{V_o}{V_i} = -K \frac{s}{s + \omega_c}$$

$$\text{where } K = \frac{R_2}{R_1} \quad \omega_c = \frac{1}{R_1 C}$$

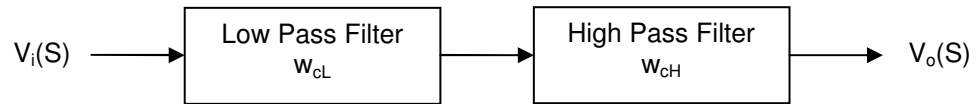


❖ Scaling

Gain scaling $\rightarrow R' = K_g R, \quad L' = K_g L, \quad C' = C/K_g$

Frequency Scaling $\rightarrow R' = R, \quad L' = L/K_f, \quad C' = C/K_f$

❖ Band Pass Filter



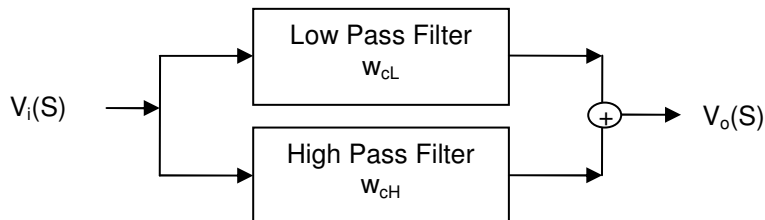
$$w_{cL} > w_{cH}$$

$$w_o = \sqrt{w_{cL} w_{cH}}$$

$$\beta = w_{cL} - w_{cH}$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\beta s}{s^2 + \beta s + w_o^2}$$

❖ Band Reject Filter



$$w_{cL} < w_{cH}$$

$$w_o = \sqrt{w_{cL} w_{cH}}$$

$$\beta = w_{cH} - w_{cL}$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s^2 + w_o^2}{s^2 + \beta s + w_o^2}$$

❖ n^{th} Order Low Pass Filter Cut Off Frequency

$$w_{C_n} = \left\{ \sqrt[n]{2} - 1 \right\} w_{C_1}$$

13.10. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 15.

13.11. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 14. Two-Port Circuit or Network

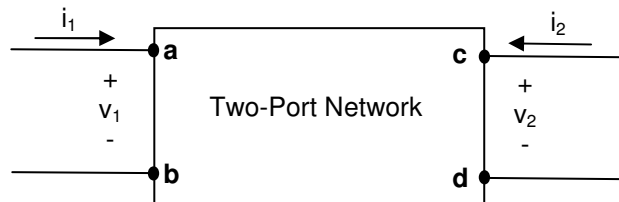
Key Concepts and Overview

- ❖ Introduction
- ❖ Two-Port Circuit Models
- ❖ Two-Port Circuit Analysis
- ❖ Interconnecting Multiple Two-Port Circuits
- ❖ Additional Resources

14.1. Introduction

As discussed in earlier chapters, it is beneficial to use models in simplifying design and analysis processes. The Two-port Network model makes it possible to use of 4 parameters to define a passive electrical circuit. The Two-port Network model may also be referred to as Two-Port Circuit model or Four-Terminal circuit. This model is used in Analysis of filters, transistors and other electrical devices.

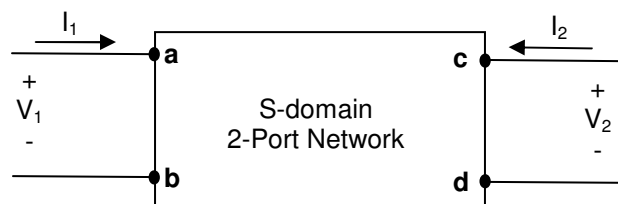
A Two-port circuit is an electrical device with two pairs of terminals. Each pair of terminals is referred to as a port which is defined by its voltage and current. Typically the two ports are input and output of the circuit. Below is a system diagram of Two-port circuit using passive convention:



The following restrictions apply to all Two-port Circuit analysis in this chapter:

- Circuit cannot include any stored energy.
- Circuit should not contain any independent sources. Although, dependent source are allowed.
- The current in and out of each port must be equal. In other words, current entering terminal "a" must be equal to current exiting terminal "b" while current entering terminal "c" must equal current exiting terminal "d".
- External connections are only made at the two ports.

Two-port Circuit is modeled based on the ports' current and voltage variables (i_1 , v_1 , i_2 , v_2) and make no use of the internal current and voltages in its external definition. Furthermore, Two-port Circuit analysis is performed in s-domain since it provides a more general analysis environment:



14.2. Two-Port Network Models

There are multiple approaches to selecting, calculating and measuring the value of two-port network parameters. The most common parameter types used to model two-port network are listed below:

- . (1) Impedance Parameters (z-Parameters)
- . (2) Admittance Parameters (y-Parameters)
- . (3) Hybrid Parameters (h-Parameters)
- . (4) Inverse Hybrid Parameters (g-Parameters)
- . (5) Transmission Parameters (ABCD-Parameters)

These parameters may be calculated if the internal circuitry is exposed for analysis otherwise they may be derived from measurement of ports' currents and voltages. Any of these parameter sets may be converted to any other by algebraic operations.

In the following sections, Two-port Network models will be introduced utilizing of the five parameter sets:

❖ Impedance Parameters (Z Parameters)

The circuit is defined by the following two equations:

$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

or in matrix form
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

where :

$$z_{11} = \frac{V_1}{I_1} \Omega \quad \text{and} \quad z_{21} = \frac{V_2}{I_1} \Omega \quad \text{when } I_2 = 0$$

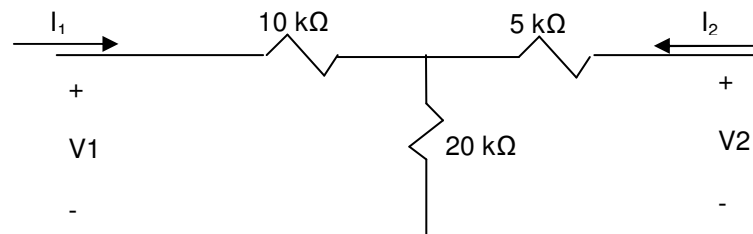
$$z_{22} = \frac{V_2}{I_2} \Omega \quad \text{and} \quad z_{12} = \frac{V_1}{I_2} \Omega \quad \text{when } I_1 = 0$$

The z parameter definitions above simply state that z_{11} and z_{21} are measured or calculated with $I_2=0$ (or port 2 is open). While, z_{22} and z_{12} are measured or calculated with $I_1=0$ (or port 1 is open).

When working with 2 port network, you are either given access to circuit drawing (White box) or just given access to the external two port (Black box).

➤ White box problem example

You are given a circuit diagram such as the one shown below and asked to write the Z parameters.



To solve this type of problem, apply circuit analysis techniques such as KCL/KVL and z parameter definitions to calculate each parameter:

$$I_1 = 0 \rightarrow Z_{12} = V_1/I_2 = 20 \text{ K}\Omega$$

$$Z_{22} = V_2/I_2 = 25 \text{ K}\Omega$$

$$I_2 = 0 \rightarrow Z_{11} = V_1/I_1 = 30 \text{ K}\Omega$$

$$Z_{21} = V_2/I_1 = 20 \text{ K}\Omega$$

$$Z = \begin{bmatrix} 30 & 20 \\ 20 & 25 \end{bmatrix} \text{ K}\Omega$$

➤ Black box problem example

In this Black box type problem, you are only given the data that can be obtained using multimeter and access to external port such as the following table:

I_1	V_1	I_2	V_2
0	1 V	2 A	3 V
1 A	2 V	0	4 V

To solve this type of problem, apply the data in the table to the Z parameter definition to calculate each parameters:

$$I_1 = 0 \rightarrow Z_{12} = V_1/I_2 = 0.5 \text{ }\Omega$$

$$Z_{22} = V_2/I_2 = 1.5 \text{ }\Omega$$

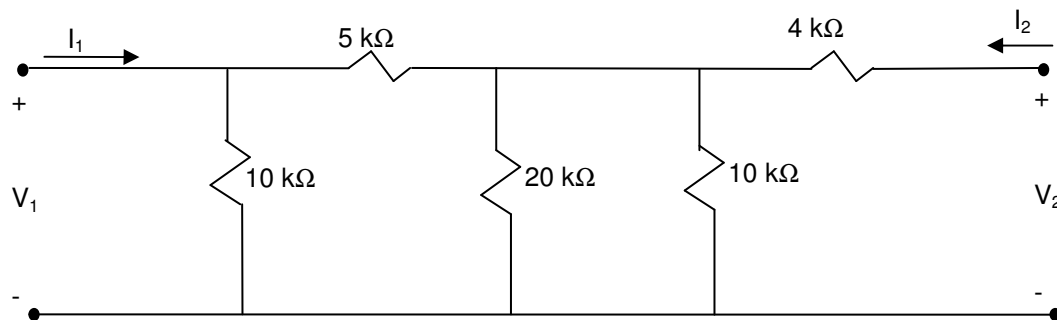
$$I_2 = 0 \rightarrow Z_{11} = V_1/I_1 = 2 \text{ }\Omega$$

$$Z_{21} = V_2/I_1 = 4 \text{ }\Omega$$

$$Z = \begin{bmatrix} 2 & 0.5 \\ 4 & 1.5 \end{bmatrix} \Omega$$

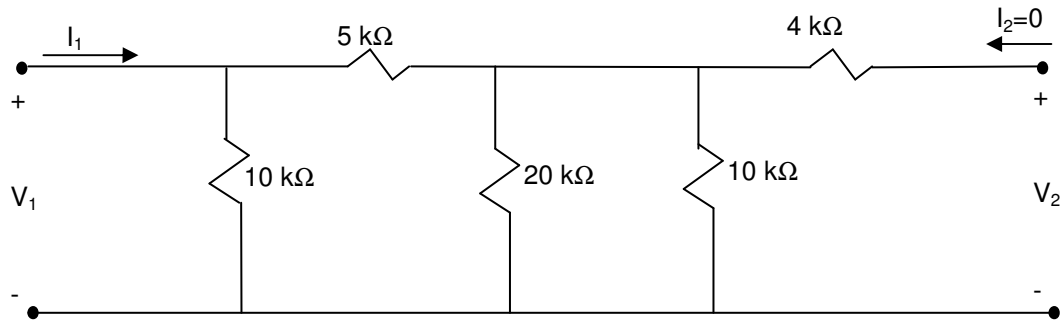
➤ Example – z-parameters

Redraw this circuit as two-parameter network and calculate the corresponding z-parameters.



Solution:

Start with $I_2 = 0$ to find $Z_{11} = V_1/I_1$ and $Z_{21} = V_2/I_1$ which is the same as the equivalent resistance of the following circuit:



Write KCL equations for V_1 and V_2 :

$$-I_1 + V_1/10,000 + (V_1 - V_2)/5,000 = 0$$

$$-I_2 + V_2/10,000 + V_2/20,000 + (V_2 - V_1)/5,000 = 0$$

Rewrite equations:

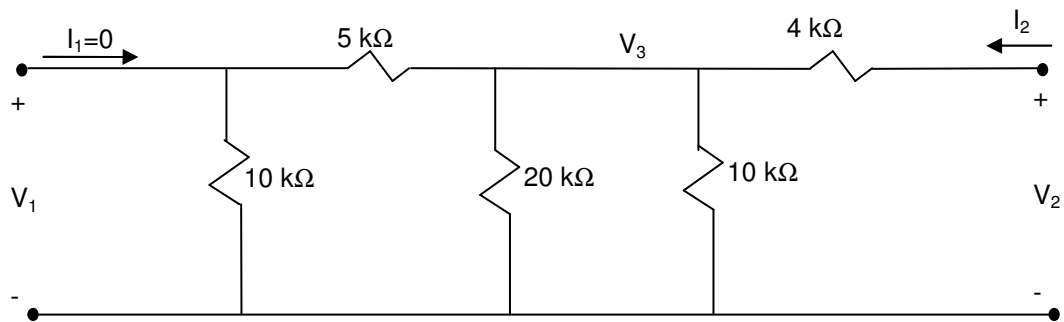
$$-10,000I_1 + 3V_1 - 2V_2 = 0$$

$$7V_2 - 4V_1 = 0 \rightarrow V_2 = (4/7)V_1$$

Substitute V_2 into the First equation $\rightarrow -10,000I_1 + (16/7)V_1 = 0 \rightarrow Z_{11} = V_1/I_1 = 5384.6 \Omega$

Substitute $V_2 \rightarrow Z_{21} = (4/7)V_1/I_1 = 3076.9 \Omega$

Next, Set $I_1 = 0$ to find $Z_{12} = V_1/I_2$ and $Z_{22} = V_2/I_2$ which is the same as the equivalent resistance of the following circuit:



Write KCL equations for V_1 and V_2 :

$$-I_1 + V_1/10,000 + (V_1 - V_3)/5,000 = 0$$

$$(V_3 - V_1)/5,000 + V_3/10,000 + V_3/20,000 + (V_3 - V_2)/4,000 = 0$$

$$-I_2 + (V_2 - V_3)/4,000 = 0$$

Rewrite equations:

$$3V_1 - 2V_3 = 0 \rightarrow V_3 = (3/2)V_1$$

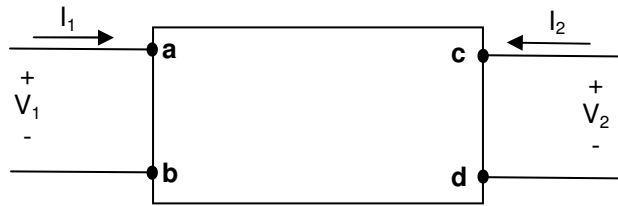
$$-4V_1 - 5V_2 + 12V_3 = 0 \rightarrow V_2 = (14/5)V_1$$

$$-4,000I_2 + V_2 - V_3 = 0 \rightarrow -4,000I_2 + (13/10)V_1 = 0 \quad Z_{12} = V_1/I_2 = 40,000/13 = 3076.9 \Omega$$

Substitute $V_1 \rightarrow Z_{22} = V_2/I_2 = (13/10)V_1/I_2 = 8615 \Omega$

Z-parameter 2 port network may be represented as:

$Z_{11} = 5384 \Omega$, $Z_{12} = 3076.9 \Omega$, $Z_{21} = 3076.9 \Omega$, $Z_{22} = 8615 \Omega$,



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 5384 & 3076.9 \\ 3076.9 & 8615 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

❖ Admittance Parameter (y-Parameters)

The circuit is defined by the following two equations:

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

or in matrix form
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where :

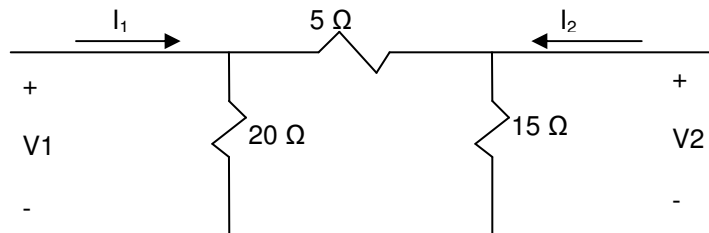
$$y_{11} = \frac{I_1}{V_1} \text{ S and } y_{21} = \frac{I_2}{V_1} \text{ S when } V_2 = 0$$

$$y_{22} = \frac{I_2}{V_2} \text{ S and } y_{12} = \frac{I_1}{V_2} \text{ S when } V_1 = 0$$

The y parameter definitions above state that y_{11} and y_{21} are measured or calculated with $V_2=0$ (or port 2 is shorted). While, y_{22} and y_{12} are measured or calculated with $V_1=0$ (or port 1 is shorted).

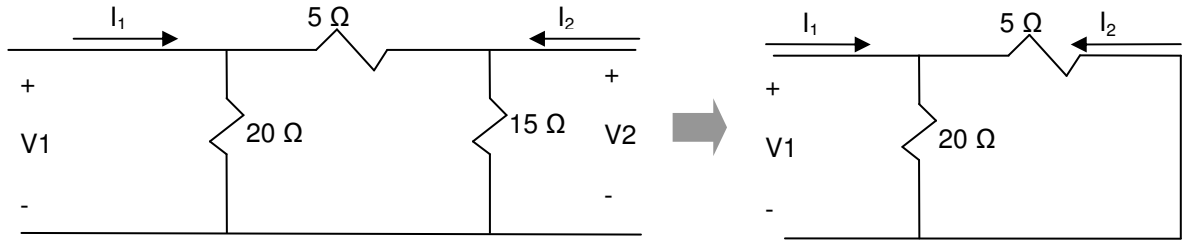
➤ Example – y-parameters

Find the y-parameters for the following circuit?



Solution:

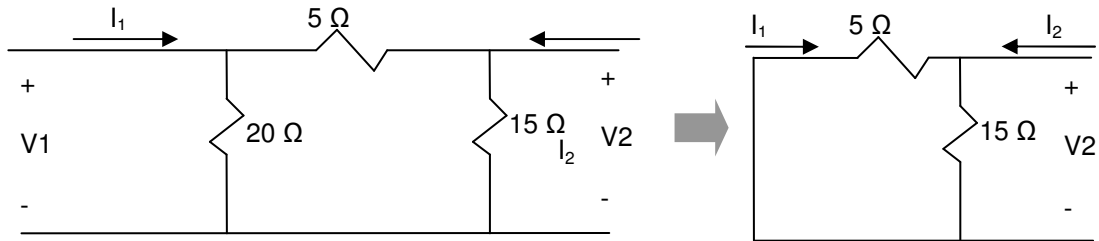
Step 1. let $V_2=0$ which means short output to find y_{11} & y_{12}



$$I_1 = V_1 / (20 \parallel 5) \rightarrow y_{11} = I_1 / V_1 = 1/4 \text{ S}$$

$$I_2 = -V_1 / (5) \rightarrow y_{21} = I_2 / V_1 = 1/5 \text{ S}$$

Step 2. Let $V_1 = 0$ which means the input is short circuited



$$I_2 = V_2 / (15 \parallel 5) \rightarrow y_{22} = I_2 / V_2 = 4/15 \text{ S}$$

$$I_1 = -V_2 / (5) \rightarrow y_{12} = I_1 / V_2 = -1/5 \text{ S}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/5 \\ 1/5 & 4/15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

❖ Hybrid Parameters (h-Parameters)

$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$I_2 = h_{21} I_1 + h_{22} V_2$$

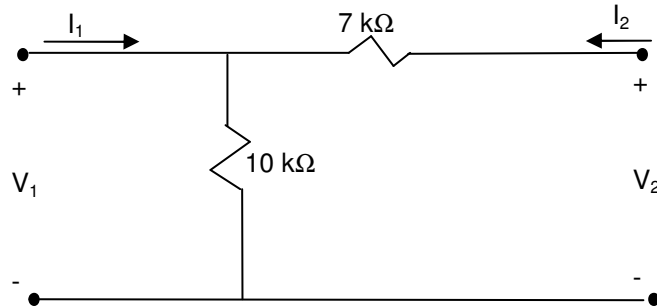
or in matrix form
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

where :

$$h_{11} = \frac{V_1}{I_1} \Omega \quad \text{and} \quad h_{21} = \frac{I_2}{I_1} \quad \text{when } V_2 = 0$$

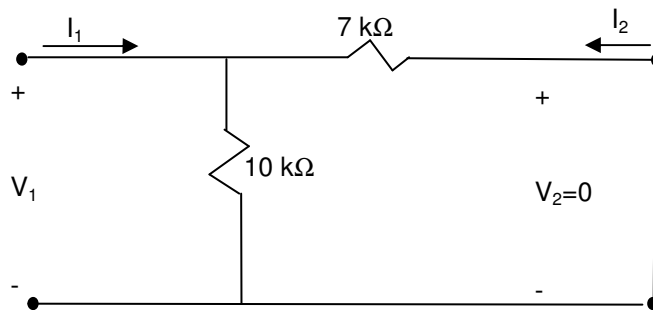
$$h_{12} = \frac{V_1}{V_2} \quad \text{and} \quad h_{22} = \frac{I_2}{V_2} \text{ S} \quad \text{when } I_1 = 0$$

- Example – h-parameters
Find the h-parameters for the following circuit?



Solution:

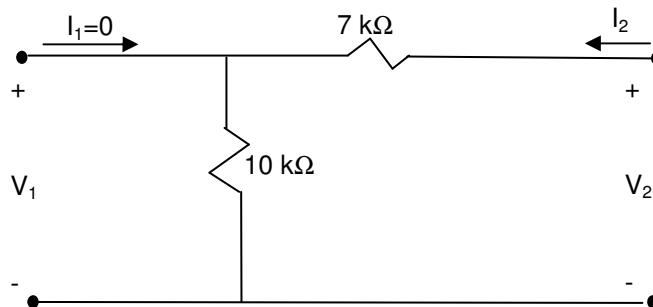
Set $V_2 = 0$ to find $h_{11} = V_1/I_1$ and $h_{21} = I_2/I_1 \rightarrow$



$$h_{11} = V_1/I_1 = R_{eq} = 1/(1/10 + 1/7) = 4.1 \, \Omega$$

$$\text{We have } 10(I_1 + I_2) = 7 I_2 \rightarrow h_{21} = I_2/I_1 = -10/3$$

Set $I_1 = 0$ to find $h_{12} = V_1/V_2$ and $h_{22} = I_2/V_2 \rightarrow$



$$h_{12} = V_1/V_2 = 10/17 \text{ (voltage divider)}$$

$$h_{22} = I_2/V_2 = 1/17 \text{ S}$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 4.1 & 10/17 \\ -10/3 & 1/17 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

❖ Inverse Hybrid Parameters (g-Parameters)

$$I_1 = g_{11}V_1 + g_{12}I_2$$

$$V_2 = g_{21}V_1 + g_{22}I_2$$

or in matrix form
$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

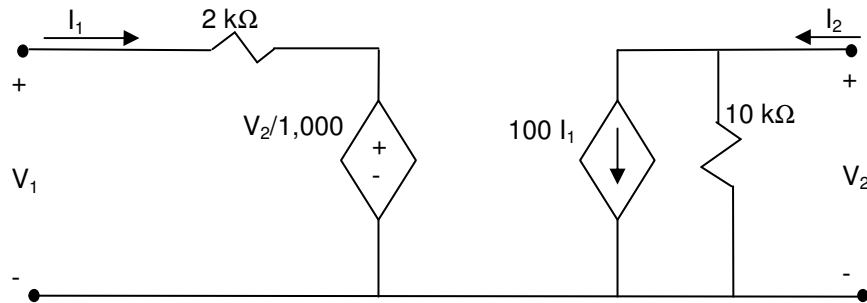
where :

$$g_{11} = \frac{I_1}{V_1} S \quad \text{and} \quad g_{21} = \frac{V_2}{V_1} \quad \text{when } I_2 = 0$$

$$g_{22} = \frac{V_2}{I_2} \Omega \quad \text{and} \quad g_{12} = \frac{I_1}{I_2} \quad \text{when } V_1 = 0$$

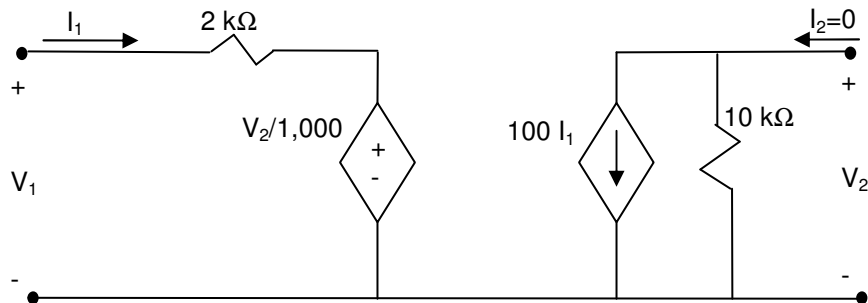
➤ Example – g-parameters

Find the g parameters for the following circuit:



Solution:

Set $I_2=0$ and find $g_{11}=I_1/V_1$ and $g_{21} = V_2/V_1$



Write KVL equations for both side→

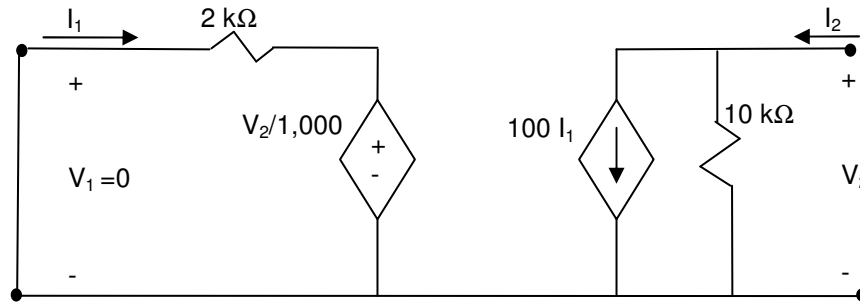
$$V_2 = -10^6 I_1$$

$$V_1 = 2,000 I_1 + V_2/1,000 \rightarrow \text{Replace } I_1 \text{ with equivalent from the above equation}$$

$$V_1 = -0.002V_2 + 0.001V_2 \rightarrow g_{21} = V_2/V_1 = -0.001$$

$$\text{Replace } V_2 \text{ in the } V_2/V_1 = -0.001 \text{ equation with } V_2 = -10^6 I_1 \rightarrow g_{11}=I_1/V_1 = 10^{-9}$$

Set $V_1=0$ and find $g_{12}=I_1/I_2$ and $g_{22} = V_2/I_2$



Write equations for both side→

$$I_2 = V_2 / 10,000 + 100 I_1$$

$$V_2 / 1000 = -2,000 I_1 \rightarrow V_2 = -2,000,000 I_1$$

Replace V_2 in first equation $\rightarrow I_2 = -200 I_1 + 100 I_1 = -100 I_1 \rightarrow g_{12} = I_1 / I_2 = -0.001$

Replace I_1 in the g_{12} equation $\rightarrow -5 \times 10^{-5} V_2 / I_2 = -0.001 \rightarrow g_{22} = V_2 / I_2 = 2000 \Omega$

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10^{-9} & -0.001 \\ -0.001 & 2,000 \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

❖ Transmission Parameters (ABCD-Parameters)

The ABCD-parameters are also known as Chain, Cascade or Transmission parameters. ABCD-parameter assumes that the current I_2 is running the opposite direction so that it would be considered positive for the next stage input in cascading two-port networks>

$$V_2 = AV_1 + BI_1$$

$$-I_2 = CV_1 + DI_1$$

or in matrix form
$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

where :

$$A = \frac{V_2}{V_1} \quad \text{and} \quad C = -\frac{I_2}{V_1} \quad \text{when } I_1 = 0$$

$$B = \frac{V_2}{I_1} \quad \text{and} \quad D = -\frac{I_2}{I_1} \quad \text{when } V_1 = 0$$

➤ Example – ABCD-parameters

Find the value of ABCD parameter if the following equations are given for the circuit:

$$25 I_1 + 30 I_2 = 4V_1$$

$$16 I_1 - 40 I_2 = 8V_2$$

Solution:

Set $I_1=0$ and rewrite the equations to find $A=V_2/V_1$ and $C = -I_2/V_1$

$$30 I_2 = 4V_1 \rightarrow C = -I_2/V_1 = -4/30 \text{ Mhos}$$

$$40 I_2 = 8V_2 \rightarrow \text{replace } I_2 \text{ with value from first eq.} \rightarrow 40 \cdot 4 \cdot V_1 / 30 = 8V_2 \rightarrow 16 V_1 = 8V_2$$

$$A = V_2/V_1 = 2$$

Set $V_1=0$ and rewrite the initial equations to find $B=V_2/I_1$ and $D = -I_2/I_1$

$$25 I_1 + 30 I_2 = 0 \rightarrow 25 I_1 = -30 I_2 \rightarrow D = -I_2/I_1 = 25/30 = 5/6$$

$$16 I_1 - 40 I_2 = 8V_2 \rightarrow \text{Replace } I_2 \text{ with its value from first eq.} \rightarrow 16 I_1 + 40 \cdot (5/6) \cdot I_1 = 8V_2$$

$$\rightarrow 296 I_1 = 48V_2 \rightarrow B = V_2/I_1 = 296/48 \, \Omega$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 2 & 296/48 \\ -4/30 & 5/6 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

14.3. Two-Port Network Analysis

As mentioned early, any of the two port parameter sets can be converted to any other by rearranging one set of the two simultaneous equations to the form for the desired parameter set equation. This is simply an exercise in algebra as shown by the example of writing h-parameter equation for a circuit that is originally described by the z-parameter equations:

$$V_1 = 25I_1 + 5I_2$$

$$V_2 = 4I_1 + 16I_2$$

We know that h-parameter equation must be of the form:

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

Let's focus on deriving the first h-parameter equation:

Derive I_2 equation from second z-parameter equation $\rightarrow I_2 = (V_2 - 4I_1)/16$

Plug this value into first z-parameter equation $\rightarrow V_1 = 25I_1 + 5(V_2 - 4I_1)/16$

Simplify to get the first h-parameter eq. $\rightarrow V_1 = 23.5I_1 + (5/16)V_2$

Now we turn our attention to deriving the second h-parameter equation:

Simply re-arrange second Z-parameter $\rightarrow I_2 = (4/16)I_1 + (1/16)V_2$

Therefore the circuit may be represented with the following h-parameter equations:

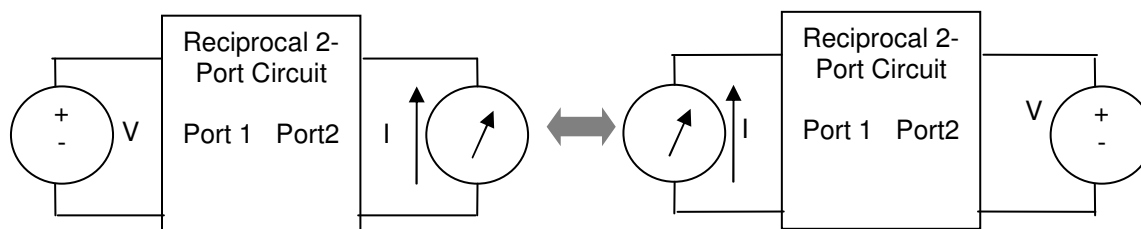
$$V_1 = 23.50I_1 + 0.31V_2$$

$$I_2 = 0.25I_1 + 0.06V_2$$

In special case 2-port networks allow us to simplify analysis. The two special cases that will be discussed here are Reciprocal and Symmetric:

❖ Reciprocal 2-port Network

A 2-port network is Reciprocal if placing an ideal voltage source on port 1 and an ideal Ammeter on port 2 will provide you with the same reading as placing the ideal voltage source on port 2 and an ideal Ammeter on port 1. In other words, current (I) will be the same in both cases:



For the Reciprocal two-port network, the following port parameter relationships exist:

$$z_{12} = z_{21}$$

$$y_{12} = y_{21}$$

$$h_{12} = -h_{21}$$

$$g_{12} = -g_{21}$$

- Student exercise – What are the relationships between ABCD parameters if the Two-port Network is a Reciprocal circuits

❖ Symmetric 2-port Network

A reciprocal 2-port network is also symmetric if its ports may be interchanged without affecting the values of current and voltage at each port. In other words, ratio of voltage and current are the same ($V_1/I_1 = V_2/I_2$).

In a symmetric 2-port network, these additional parameter relationships exist:

$$z_{11} = z_{22}$$

$$y_{11} = y_{22}$$

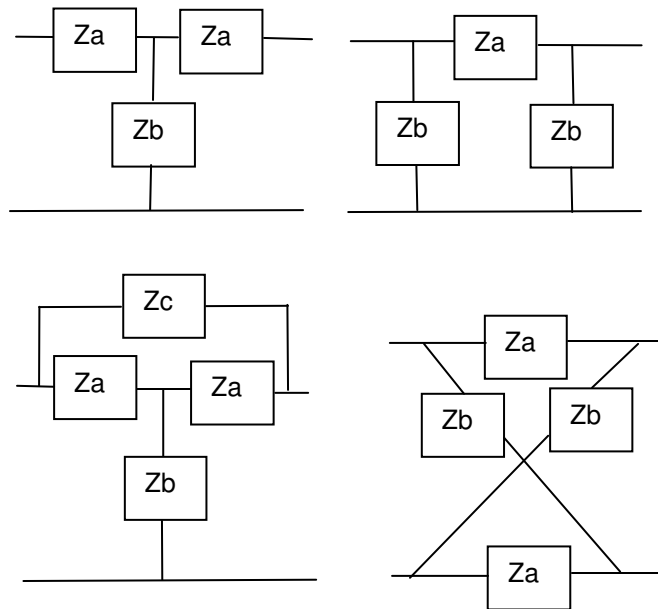
$$\Delta h = h_{11}h_{22} - h_{12}h_{21} = 1$$

$$\Delta g = g_{11}g_{22} - g_{12}g_{21} = 1$$

- Student exercise – What are the additional relationships between ABCD parameters when the Two-port Network is Symmetric circuit in addition to being a Reciprocal circuits

Here are four example of Symmetric network:

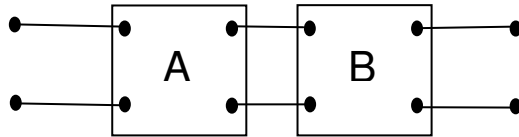
Note: By definition these examples are also Reciprocal.



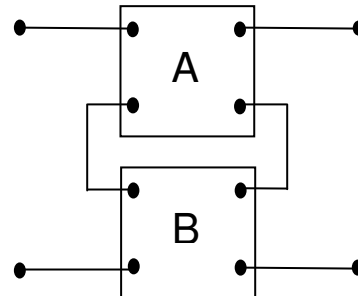
As seen here, by recognizing Reciprocal and Symmetric network, the parameter calculation is significantly simplified.

14.4. Connecting Multiple Two-Port Circuits

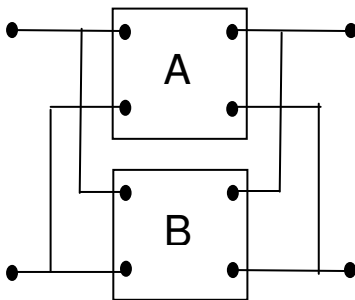
Two port parameters may be connected in five different connection patterns such as cascade, series, parallel and series-parallel and parallel series. The following diagrams show each of the connection patterns:



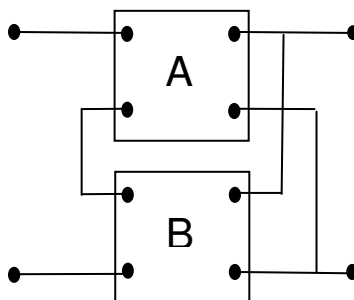
(1) Cascade



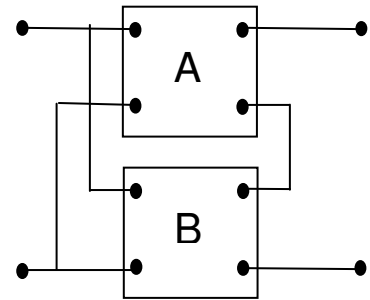
(2) Series



(3) Parallel



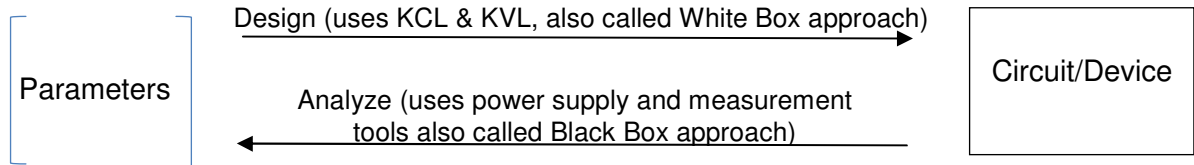
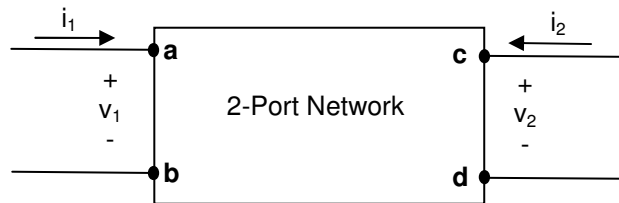
(4) Series-Parallel



(3) Parallel-Series

The parameter equations may be used to derive two port equations for each of the configuration shown above.

14.5. Summary



Five common 2-port network parameters are listed as system of equations below:

- ❖ Impedance Parameters (z-Parameters)

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

- ❖ Admittance Parameters (y-Parameters)

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

- ❖ Hybrid Parameters (h-Parameters)

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

- ❖ Inverse Hybrid Parameters (g-Parameters)

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

- ❖ Transmission Parameters (ABCD-Parameters)

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

14.6. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapter 18.
- ❖ *S-Parameter is a generalized form of the discussion in the two-port network and it is also the basis for high frequency measurement tools. For more information on the topic refer to:*
 - Pozar, D. "Microwave Engineering" (2005) Wiley & Sons
Pages 170-185.
 - S-Parameter Application note "AN 154" by Agilent
<http://cp.literature.agilent.com/litweb/pdf/5952-1087.pdf>

14.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 15. Electromagnetic Coupling

Key Concepts and Overview

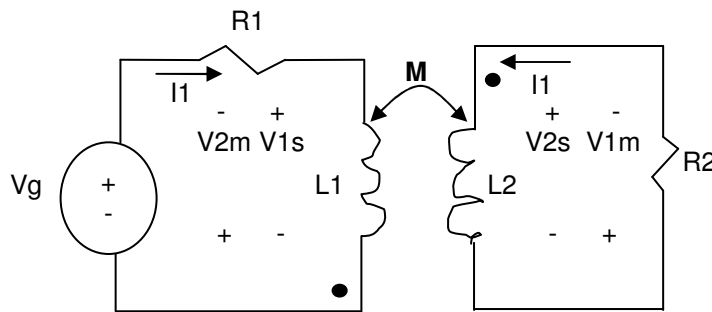
- ❖ Introduction
- ❖ Mutual Inductance
- ❖ Transformers
- ❖ AC Motors
- ❖ DC Motors
- ❖ Additional Resources

15.1. Introduction

❖ TBC

15.2. Mutual Inductance

- So far we have talk only about circuit elements that are linked directly. Now we need to introduce the case where the magnetic field generated by energy in one element creates current in a second element.



Mesh Current Analysis:

KCV for Mesh 1

$$-v_g + i_1 R_1 + (L_1 di_1/dt - M di_2/dt) = 0$$

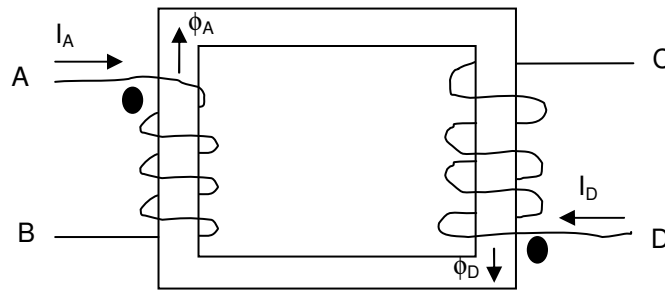
KCV for Mesh 2

$$i_2 R_2 + (L_2 di_2/dt - M di_1/dt) = 0$$

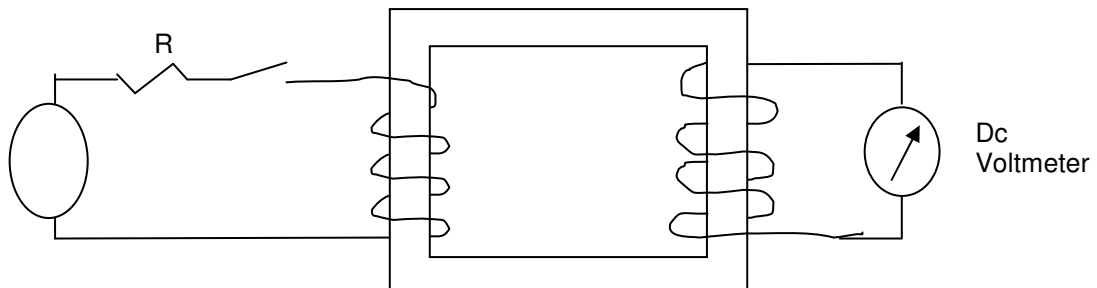
- * Magnetically Coupled Coils (M stands for Mutual Inductance)
- * The curved arrow show which two elements are coupled
- Here the two circuit are linked by magnetic field
 - 1) Time Varying current in one circuit induces voltage in a second circuit.
 - 2) The voltage in second circuit is related to current in the first circuit by the mutual inductance (M)
- Mesh current analysis is the best way to analyze circuits containing Mutual Inductance
 - 1) Step 1: Pick a reference direction for current through each coil.
 - 2) Step 2: Sum the voltage around each mesh (KVL)

Note there are two voltages across each coil (for example in L1)

 - (1) Self induced voltage $L_1 di_1/dt$
 - (a) Sign is + where the current enters the inductor
 - (2) Mutually induced voltage $M di_2/dt$ (see the current is from the other coil)
 - (a) Sign on mutual inductance depend on the winding direction. In order to simplify the process a dot convention is used.
 - (b) When the reference current enters the dotted terminal of a coil then the reference polarity of the voltage that it induces in the other coil is positive at its dotted terminal.
OR
when the reference direction for a current leaves the dotted terminal of a coil, the reference polarity of the voltage that it induced in the other coil is negative at its dotted terminal.
- Example – To Be Added
- The procedure for determining the Dot Marking
 - 1) Physical construction is known– right hand rule and flux has to be in same direction. The six steps:
 - 1) Arbitrarily select one terminal (For example D and mark it with a dot)
 - 2) Assign a current into the dotted terminal (I_d)
 - 3) Use the right hand rule (wrap your hand in the direction of current, then your thumb will pointing in the direction of created flux (Magnetic field inside the core, ϕ_d)
 - 4) Arbitrarily select one terminal on the second coil and assign a current into the dotted terminal (I_a)
 - 5) Use the right hand rule (wrap your hand in the direction of current, then your thumb will pointing in the direction of created flux (Magnetic field inside the core, ϕ_a)
 - 6) If both fluxes are in the same direction then place a dot on the terminal that current enters. Otherwise place a dot on the terminal that current leaves.



- Relative marking when Physical construction is not known
This is done through experimentation using the following steps
- 1) Connect DC voltage source, a resistor, a switch in series to one winding
 - 2) Connect Voltmeter to the other winding
 - 3) Mark the terminal connect to the positive side of DC voltage source with a dot
 - 4) If the momentary deflection is upscale, the coil terminal connected to positive terminal of the voltmeter should be marked with a dot. Otherwise, mark the other terminal.

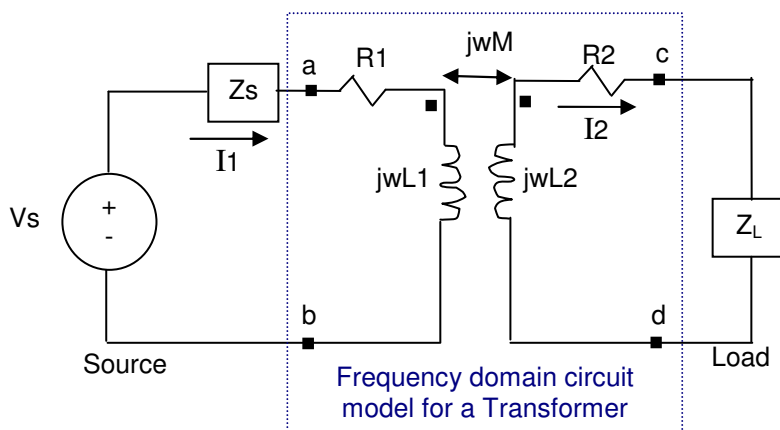


15.3. Transformers

❖ The Transformer

- Linear Transformer (two magnetically coupled winding with a single core) use in Communication Circuits to:
 - Match impedance
 - Eliminate DC signals
- Ideal Transformer (Two magnetically coupled windings, each around a different core) used in Power systems

Observation: When dealing with Mutual inductance, mesh-current method is much less cumbersome than Node-voltage method.



❖ Linear Transformer

- A Linear transformer is created when two coils are wound on a single core to ensure magnetic coupling.
 - Source-side winding is called Primary Winding
 - (1) R_1 = the resistance of the primary winding
 - (2) L_1 = the self-inductance of the primary winding
 - Load-side winding is called Secondary Winding
 - (1) R_2 = the resistance of the secondary winding
 - (2) L_2 = the self-inductance of the secondary winding
 - M = the mutual inductance = $K\sqrt{L_1 L_2}$ where K is coefficient of coupling.
 - Z_{11} = Primary Self inductance = $Z_s + R_1 + jwL_1$
 - Z_{22} = Secondary Self inductance = $R_2 + jwL_2 + Z_L$
- Write the mesh for primary and secondary loop to relate various parameters
 - $V_s = (Z_s + R_1 + jwL_1)I_1 - jwMI_2 = Z_{11}I_1 - jwMI_2$
 - $0 = (Z_L + R_2 + jwL_2)I_2 - jwMI_1 = Z_{22}I_2 - jwMI_1$
 - Impedance seen by source = $Z_{int} = V_s/I_1 = Z_{11} + \frac{w^2 M^2}{Z_{22}}$
 - Impedance at "ab" port = $Z_{ab} = Z_{int} - Z_s = R_1 + jwL_1 + \frac{w^2 M^2}{R_2 + jwL_2 + Z_L}$
- Note the third term is purely due to the secondary winding being reflected back
 - Reflected impedance = $Z_r = \frac{w^2 M^2}{R_2 + jwL_2 + Z_L}$
 - When $Z_L = R_L + jX_L$
 - Then $Z_r = \frac{w^2 M^2}{\{R_2 + R_L + j(wL_2 + X_L)\}}$

Multiply top and bottom by the conjugate and simplify

$$Z_r = \{w^2 M^2 / |Z_{22}|^2\} \{(R_2 + R_L) - j(wL_2 + X_L)\}$$

- Notice that Z_r is a scalar value multiplied by the conjugate of $Z_{22} = (R_2 + R_L) + j(wL_2 + X_L)$

❖ Example – To Be Added.

❖ Ideal Transformer (Ferro Magnetic core material is closest to ideal transformer due to high permanence)

➤ Ideal Transformer consists of two magnetically coupled coils & core having the following characteristics:

- N_1 turn in primary winding and N_2 turns in secondary winding
- Unity coefficient of coupling ($k=1$) which means $M = \sqrt{L_1 L_2}$
- Infinite self inductance ($L_1 \rightarrow \infty$ & $L_2 \rightarrow \infty$)
- Negligible coil loss due to parasitic resistance ($R_1 = R_2 = 0$)

➤ Applying the ideal characteristic we will arrive at:

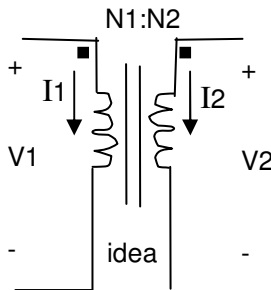
- Voltage $\rightarrow \frac{V_1}{V_2} = \pm \frac{N_1}{N_2}$

If the voltage polarities are the same at the dots use minus sign other wise use plus.

- Current $\rightarrow \frac{I_1}{I_2} = \pm \frac{N_2}{N_1}$

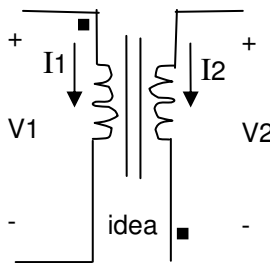
If the current flows are both in or out of the dots use minus sign otherwise use plus.

Examples:



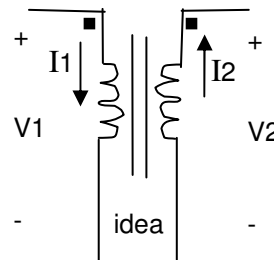
$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$$\frac{I_1}{I_2} = -\frac{N_2}{N_1}$$



$$\frac{V_1}{V_2} = -\frac{N_1}{N_2}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

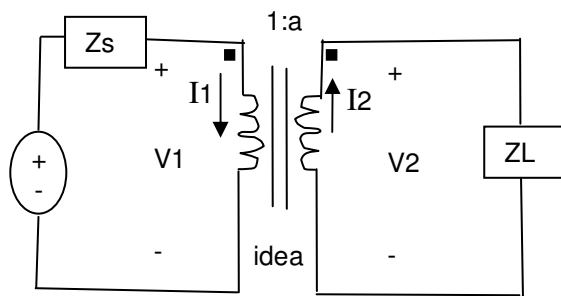


$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

❖ Using an ideal transformer for Impedance Matching

➤ Let $a = N_2/N_1$ be the scaling factor



$$V_1/V_2 = 1/a \rightarrow V_1 = V_2/a \quad \& \quad I_1/I_2 = a \rightarrow I_1 = aI_2$$

Then $Z_{in} = V_1/I_1 = (1/a^2)(V_2/I_2)$ or

$Z_{in} = (1/a^2)Z_L$ which means The ideal transformer's secondary coil reflects the load impedance back to the primary coil, with scaling factor $(1/a^2)$

❖ Example – TBC

15.4. DC Motors

❖ TBC

15.5. AC Motors

❖ TBC

15.6. Summary

15.7. Additional Resources

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
Chapters 6, 7 & 8.
- ❖ Matsch, L. *Electromagnetic & Electromechanical Machines* (1977) IEP.
Chapters 1, 3, 4 & 5

15.6. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Appendix A. Resistor Band Color Definition

Resistor Value Color Bands

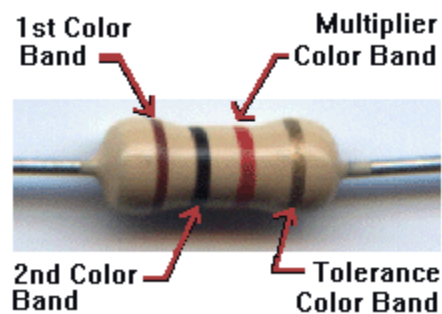
<i>Band Colors</i>	<i>Value Bands, 1st & 2nd</i>	<i>Multiplier Color Band, 3rd</i>
BLACK	0	x1
BROWN	1	x10
RED	2	x100
ORANGE	3	x1,000 or 1K
YELLOW	4	x10,000 or 10K
GREEN	5	x100,000 or 100K
BLUE	6	x1,000,000 or 1M
VIOLET	7	x10,000,000 or 10M
GRAY	8	x100,000,000 or 100M
WHITE	9	x1000,000,000 or 1G

Note: If third band is gold then divide by 10 and if silver divide by 100.

Resistor Tolerance Color Bands

<i>Band Colors</i>	<i>Tolerance Color Band, 4th</i>
GOLD	5%
SILVER	10%
NONE	20%

Example (1.2 K Ω or 1200 Ω resistor)



Appendix B. LM 324 OpAmp

The LM324 series is a low-cost, quad operational amplifier with true differential inputs. They are the most commonly used general low power and low frequency Op Amps. The usage is limited to low frequency (less than 10 Hz) since the gain changes as the frequency changes. For many applications, it may be used as long as it is understood that open loop gain at higher frequency will be lower. One of its major advantages is that it only requires a single supply for operation. LM 324 operate with supply voltage ranging as low as 3.0 V or as high as 32 V.

It may also be useful to know that LM 324 DC parameters are as follows:

- Open Loop Gain $A = 100,000$
- Output Resistance $R_o = 75\ \Omega$
- Input Resistance $R_i = 2\ \text{M}\Omega$

For a complete data sheet on LM 324 and other Op Amps, refer to the course website.

Appendix C. Additional Resources & Future Improvements

- ❖ Additional resources are available on the course website <http://www.EngrCS.com>.
- ❖ Submit all your improvement ideas to www.EngrCS.com